V. L. MIRONOV, S. V. MIRONOV

SPACE - TIME SEDEONS AND THEIR APPLICATION IN RELATIVISTIC QUANTUM MECHANICS AND FIELD THEORY

$$\mathbf{e}_t \mathbf{e}_r = i\, \mathbf{e}_{tr}$$

Institute for physics of microstructures RAS
Nizhny Novgorod 2014

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Dedicated to Galina Mironova,

wife and mother

Content

Foreword

This book is a systematic exposition of the algebra of sixteen-component space-time values "sedeons" and their applications to describe quantum particles and fields. The book contains the results of our several articles published in the period 2008-2014. It includes a large number of carefully selected reference material relating to the use of different multi-component algebras in physical problems and may be useful as an introduction to the application of hypercomplex numbers in physics.

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Introduction

One of the first non-commutative multi-component algebras the algebra of four-component quaternions was discovered in 1843 by W.R.Hamilton [1,2]. Substantially the quaternions are a generalization of complex numbers on a space of dimension 4. Then J.Graves (1843) and independently A.Cayley (1845) discovered the eight-component values octonions [3]. An algorithm for constructing octonions on the basis of quaternions is called as Cayley-Dickson construction procedure. It enables the generalization of the complex numbers on the any space of dimension 2^n and in particular, to build the sixteen-component hypercomplex numbers sedenions [4]. The history of discovery of hypercomplex numbers partially considered in [3,5]. A systematic exposition of the theory of quaternions and hypercomplex algebras of higher dimension can be found in the following books [6-11]. An extensive bibliography on the use of quaternions in physics is contained in the reviews [12,13].

A significant disadvantage of hypercomplex numbers algebras of dimension greater than 4 is their nonassociativity. This considerably complicates their application to the description of physical systems, since all equations have to fix a specific sequence of actions of all operators. However, the Cayley-Dickson hypercomplex numbers are not only dedicated algebraic system on which one can build a description of physical systems. There are other alternative approaches based on the use of associative algebras of multi-component vectors and Clifford algebras [14,15].

This book provides a systematic presentation of the authors proposed an associative algebra of sixteen-component space-time variables "sedeons" and their applications to the description of quantum particles and fields [16-19].

Chapter 1. Algebra of sedeons

From a mathematical point of view, one of the main problems addressed in this book is the problem of representation of quadratic forms

$$\sum_{k=1}^{N} A_k^2 \qquad (1.1.)$$

as the product of two factors. In general, quadratic form (1.1.) can be expressed as follows:

$$\begin{aligned}&\left(A_1^2 + A_2^2 + ... + A_k^2 + ...A_N^2\right)\\&= \left(\alpha_1 A_1 + ... + \alpha_k A_k + ... + \alpha_N A_N\right)\left(\alpha_1 A_1 + ... + \alpha_k A_k + ... + \alpha_N A_N\right).\end{aligned} \qquad (1.2)$$

This representation is possible for two different systems of coefficients α_k. The first case corresponds to a non-commutative α_k, which have the following properties:

$$\begin{aligned}&\alpha_k \alpha_k = 1,\\&\alpha_k \alpha_l = -\alpha_k \alpha_l \text{ (for } k \neq l).\end{aligned} \qquad (1.3)$$

The second case corresponds to the orthogonal α_k:

$$\begin{aligned}&\alpha_k \alpha_k = 1,\\&\alpha_k \alpha_l = 0 \text{ (for } k \neq l).\end{aligned} \qquad (1.4)$$

In this book, both of these approaches are used to describe space-time and charge properties of physical systems. The main tool for the description we chosen the algebra of space-time sedeons.

A key feature of the sedeonic algebra and its main difference from widespread Gibbs-Heaviside vector algebra is the concept of Clifford product of vectors. Let us consider two arbitrary vectors $\vec{A}$ and $\vec{B}$ recorded in the basis of unit vectors $\vec{\boldsymbol{i}}_1$, $\vec{\boldsymbol{i}}_2$, $\vec{\boldsymbol{i}}_3$:

$$\begin{aligned}&\vec{A} = A_1\vec{\boldsymbol{i}}_1 + A_2\vec{\boldsymbol{i}}_2 + A_3\vec{\boldsymbol{i}}_3,\\&\vec{B} = B_1\vec{\boldsymbol{i}}_1 + B_2\vec{\boldsymbol{i}}_2 + B_3\vec{\boldsymbol{i}}_3.\end{aligned} \qquad (1.5)$$

Then the Clifford product for $\vec{A}$ and $\vec{B}$ is the direct product written in the following form:

$$
\begin{aligned}
\vec{A}\vec{B} &= \left(A_1\vec{\boldsymbol{i}}_1 + A_2\vec{\boldsymbol{i}}_2 + A_3\vec{\boldsymbol{i}}_3\right)\left(B_1\vec{\boldsymbol{i}}_1 + B_2\vec{\boldsymbol{i}}_2 + B_3\vec{\boldsymbol{i}}_3\right) \\
&= A_1B_1\vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_1 + A_2B_2\vec{\boldsymbol{i}}_2\vec{\boldsymbol{i}}_2 + A_3B_3\vec{\boldsymbol{i}}_3\vec{\boldsymbol{i}}_3 \\
&+ A_1B_2\vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_2 + A_2B_3\vec{\boldsymbol{i}}_2\vec{\boldsymbol{i}}_3 + A_3B_1\vec{\boldsymbol{i}}_3\vec{\boldsymbol{i}}_1 \\
&+ A_1B_3\vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_3 + A_2B_1\vec{\boldsymbol{i}}_2\vec{\boldsymbol{i}}_1 + A_3B_2\vec{\boldsymbol{i}}_3\vec{\boldsymbol{i}}_2 \ .
\end{aligned} \tag{1.6}
$$

Depending on the rules of the basis elements multiplication and commutation the Clifford product can have a different result. In particular, if we accept the rules of multiplication of the unit vectors corresponding to the Gibbs - Heaviside vector algebra

$$
\begin{aligned}
\vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_1 &= \vec{\boldsymbol{i}}_2\vec{\boldsymbol{i}}_2 = \vec{\boldsymbol{i}}_3\vec{\boldsymbol{i}}_3 = 1 \ , \\
\vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_2 &= -\vec{\boldsymbol{i}}_2\vec{\boldsymbol{i}}_1 = \vec{\boldsymbol{i}}_3 \ , \\
\vec{\boldsymbol{i}}_2\vec{\boldsymbol{i}}_3 &= -\vec{\boldsymbol{i}}_3\vec{\boldsymbol{i}}_2 = \vec{\boldsymbol{i}}_1 \ , \\
\vec{\boldsymbol{i}}_3\vec{\boldsymbol{i}}_1 &= -\vec{\boldsymbol{i}}_3\vec{\boldsymbol{i}}_1 = \vec{\boldsymbol{i}}_2 \ ,
\end{aligned} \tag{1.7}
$$

then Clifford product of two vectors $\vec{A}$ and $\vec{B}$ is equal

$$
\begin{aligned}
\vec{A}\vec{B} &= A_1B_1 + A_2B_2 + A_3B_3 \\
&+ A_2B_3\vec{\boldsymbol{i}}_1 + A_3B_1\vec{\boldsymbol{i}}_2 + A_1B_2\vec{\boldsymbol{i}}_3 \\
&- A_3B_2\vec{\boldsymbol{i}}_1 - A_1B_3\vec{\boldsymbol{i}}_2 - A_2B_1\vec{\boldsymbol{i}}_3 \ ,
\end{aligned} \tag{1.8}
$$

i.e. it is the sum of the scalar and vector products. Such approach allows to carry out the simultaneous calculations with scalar and vector quantities and is particularly fruitful in the application to the relativistic physics. However, the multiplication rules taken in vector algebra have one essential deficiency. For example, let us consider the Clifford square of the unit vector $\vec{\boldsymbol{i}}_3$. Following the rules of vector algebra (1.7), Clifford square of this vector can be represented as follows:

$$
\vec{\boldsymbol{i}}_3^{\,2} = \vec{\boldsymbol{i}}_3\vec{\boldsymbol{i}}_3 = \vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_2\vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_2 = -\vec{\boldsymbol{i}}_2\vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_1\vec{\boldsymbol{i}}_2 = -1 \ , \tag{1.9}
$$

which is in contradiction with the original rules (1.7). To overcome this contradiction it is required the development of an alternative algebra, based on other rules of multiplication.

1.1. Space-time sedeons

The sedeonic algebra [16] encloses four groups of values, which are differed with respect to spatial and time inversion.

- Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
- Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
- Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
- Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Here indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{V}}$, which is defined by the following expression:

$$\tilde{\mathbf{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}} . \qquad (1.10)$$

Let us introduce scalar-vector basis $\mathbf{a_0}$, $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, where the element $\mathbf{a_0}$ is an absolute scalar unit ($\mathbf{a_0} \equiv 1$), and the values $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ are absolute unit vectors generating the right Cartesian basis. Further we will indicate the absolute unit vectors by symbols without arrows as $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$. We also introduce the four space-time units $\mathbf{e_0}$, $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$, where $\mathbf{e_0}$ is an absolute scalar unit ($\mathbf{e_0} \equiv 1$); $\mathbf{e_1}$ is a time scalar unit ($\mathbf{e_1} \equiv \mathbf{e_t}$); $\mathbf{e_2}$ is a space scalar unit ($\mathbf{e_2} \equiv \mathbf{e_r}$); $\mathbf{e_3}$ is a space-time scalar unit ($\mathbf{e_3} \equiv \mathbf{e_{tr}}$). Using space-time basis $\mathbf{e}_\alpha$ and scalar-vector basis $\mathbf{a}_\beta$ (Greek indexes $\alpha, \beta = 0, 1, 2, 3$), we can introduce unified sedeonic components $V_{\alpha\beta}$ in accordance with following relations:

$$\begin{aligned}
V &= \mathbf{e_0} V_{00} \mathbf{a_0} , \\
\vec{V} &= \mathbf{e_0} \left(V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right) , \\
V_{\mathbf{t}} &= \mathbf{e_1} V_{10} \mathbf{a_0} , \\
\vec{V}_{\mathbf{t}} &= \mathbf{e_1} \left(V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right) , \qquad (1.11) \\
V_{\mathbf{r}} &= \mathbf{e_2} V_{20} \mathbf{a_0} ,
\end{aligned}$$

$$\vec{V}_{\mathbf{r}} = \mathbf{e_2}\left(V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3}\right),$$
$$V_{\mathbf{tr}} = \mathbf{e_3} V_{30}\mathbf{a_0},$$
$$\vec{V}_{\mathbf{tr}} = \mathbf{e_3}\left(V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3}\right).$$

Then sedeon (1.10) can be written in the following expanded form:

$$\begin{aligned}\tilde{\mathbf{V}} = \mathbf{e_0}\left(V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3}\right)\\ +\mathbf{e_1}\left(V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3}\right)\\ +\mathbf{e_2}\left(V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3}\right)\\ +\mathbf{e_3}\left(V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3}\right).\end{aligned} \tag{1.12}$$

The sedeonic components $V_{\alpha\beta}$ are numbers (complex in general). Further we will use symbol 1 instead units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity.

The important property of sedeons is that the equality of two sedeons means the equality of all sixteen space-time scalar-vector components. It enables to write many relations of modern relativistic physics in a compact form.

Let us consider the multiplication rules for basic elements $\mathbf{a_n}$ and $\mathbf{e_m}$ (Latin indexes **n**, **m** = 1, 2, 3). We require that square of the length of any vector should be positively defined quantity. Then the vectors $\mathbf{a_n}$ should satisfy the following rules:

$$\mathbf{a_n}\mathbf{a_n} = \mathbf{a_n}^2 = 1, \tag{1.13}$$

$$\mathbf{a_n}\mathbf{a_m} = -\,\mathbf{a_m}\mathbf{a_n} \text{ (for } \mathbf{n} \neq \mathbf{m}\text{)}. \tag{1.14}$$

Besides, for the existence of Clifford product we have to require the following rules of multiplication of the basis elements $\mathbf{a_n}$ (external product):

$$\mathbf{a_1}\mathbf{a_2} = i\mathbf{a_3}, \quad \mathbf{a_2}\mathbf{a_3} = i\mathbf{a_1}, \quad \mathbf{a_3}\mathbf{a_1} = i\mathbf{a_2}. \tag{1.15}$$

We introduce the similar rules for the elements of space-time basis $\mathbf{e_m}$:

$$\mathbf{e_m}\mathbf{e_m} = \mathbf{e_m}^2 = 1, \tag{1.16}$$

$$\mathbf{e_n}\mathbf{e_m} = -\,\mathbf{e_m}\mathbf{e_n} \text{ (for } \mathbf{n} \neq \mathbf{m}\text{)}, \tag{1.17}$$

$$\mathbf{e_1}\mathbf{e_2} = i\mathbf{e_3}, \quad \mathbf{e_2}\mathbf{e_3} = i\mathbf{e_1}, \quad \mathbf{e_3}\mathbf{e_1} = i\mathbf{e_2}. \tag{1.18}$$

Here and further the value i is imaginary unit $(i^2 = -1)$. The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a_n}$ and space-time units $\mathbf{e_m}$ can be presented for obviousness as the tables 1 and 2.

Table 1. Multiplication rules for absolute unit vectors.

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	1	$i\mathbf{a_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	1

Table 2. Multiplication rules for space-time units.

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_1}$	1	$i\mathbf{e_3}$	$-i\mathbf{e_2}$
$\mathbf{e_2}$	$-i\mathbf{e_3}$	1	$i\mathbf{e_1}$
$\mathbf{e_3}$	$i\mathbf{e_2}$	$-i\mathbf{e_1}$	1

Note that although sedeon units $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$, and the unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ generate anticommutative algebras

$$\mathbf{e_n e_m} = -\mathbf{e_m e_n} ,$$

$$\mathbf{a_n a_m} = -\mathbf{a_m a_n} ,$$

units $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ commute with vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$:

$$\mathbf{a_n e_m} = \mathbf{e_m a_n} \tag{1.19}$$

for any $\mathbf{n}$ and $\mathbf{m}$.

Thus the sedeon $\tilde{\mathbf{V}}$ is the complicated space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

A sedeon can be represented in compact form. Introducing the scalar-vector values as

$$\bar{\mathbf{V}}_\mathbf{0} = V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} ,$$

$$\bar{\mathbf{V}}_\mathbf{1} = V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} , \tag{1.20}$$

$$\bar{\mathbf{V}}_2 = V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3},$$
$$\bar{\mathbf{V}}_3 = V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3},$$

we can write the sedeon (1.12) in the following form:

$$\tilde{\mathbf{V}} = \bar{\mathbf{V}}_0 + \mathbf{e}_1\bar{\mathbf{V}}_1 + \mathbf{e}_2\bar{\mathbf{V}}_2 + \mathbf{e}_3\bar{\mathbf{V}}_3. \tag{1.21}$$

On the other hand, introducing the designations of space-time sedeon-scalars as

$$\begin{aligned}
\mathbf{V_0} &= V_{00}\mathbf{a_0} + \mathbf{e_1}V_{10} + \mathbf{e}_2V_{20} + \mathbf{e}_3V_{30},\\
\mathbf{V_1} &= V_{01}\mathbf{a_0} + \mathbf{e_1}V_{11} + \mathbf{e}_2V_{21} + \mathbf{e}_3V_{31},\\
\mathbf{V_2} &= V_{02}\mathbf{a_0} + \mathbf{e_1}V_{12} + \mathbf{e}_2V_{22} + \mathbf{e}_3V_{32},\\
\mathbf{V_3} &= V_{03}\mathbf{a_0} + \mathbf{e_1}V_{13} + \mathbf{e}_2V_{23} + \mathbf{e}_3V_{33},
\end{aligned} \tag{1.22}$$

we can write the sedeon (1.12) in another form

$$\tilde{\mathbf{V}} = \mathbf{V_0} + \mathbf{V}_1\mathbf{a}_1 + \mathbf{V}_2\mathbf{a}_2 + \mathbf{V}_3\mathbf{a}_3, \tag{1.23}$$

or introducing the sedeon-vector

$$\vec{\mathbf{V}} = \vec{V} + \vec{V}_{\mathbf{t}} + \vec{V}_{\mathbf{r}} + \vec{V}_{\mathbf{tr}} = \mathbf{V}_1\mathbf{a}_1 + \mathbf{V}_2\mathbf{a}_2 + \mathbf{V}_3\mathbf{a}_3, \tag{1.24}$$

it can be represented in the following compact form:

$$\tilde{\mathbf{V}} = \mathbf{V_0} + \vec{\mathbf{V}}. \tag{1.25}$$

Further we will indicate the sedeon-scalars and the sedeon-vectors with the bold capital letters.

Let us consider the sedeonic multiplication in detail. The sedeonic product of two sedeons $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ can be presented in the following form:

$$\tilde{\mathbf{A}}\tilde{\mathbf{B}} = \left(\mathbf{A_0} + \vec{\mathbf{A}}\right)\left(\mathbf{B_0} + \vec{\mathbf{B}}\right) = \mathbf{A_0}\mathbf{B_0} + \mathbf{A_0}\vec{\mathbf{B}} + \vec{\mathbf{A}}\mathbf{B_0} + \left(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}\right) + \left[\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right]. \tag{1.26}$$

Here we denote the sedeonic scalar multiplication of two sedeon-vectors (internal product) by symbol " $\cdot$ " and round brackets

$$\left(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}\right) = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3, \tag{1.27}$$

and sedeonic vector multiplication (external product) by symbol " $\times$ " and square brackets

$$\left[\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right] = i\left(\mathbf{A}_2\mathbf{B}_3 - \mathbf{A}_3\mathbf{B}_2\right) + i\left(\mathbf{A}_3\mathbf{B}_1 - \mathbf{A}_1\mathbf{B}_3\right) + i\left(\mathbf{A}_1\mathbf{B}_2 - \mathbf{A}_2\mathbf{B}_1\right). \tag{1.28}$$

In expressions (1.27) and (1.28) the multiplication of sedeonic components is performed in accordance with (1.22) and Table 2. Note that in sedeonic algebra the expression for the vector product differs from analogous expression in Gibbs vector algebra. As a consequence, in sedeonic algebra the formula for the vector triple product of three absolute vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ has the following form:

$$\left[\vec{A}\times\left[\vec{B}\times\vec{C}\right]\right] = -\vec{B}\left(\vec{A}\cdot\vec{C}\right)+\vec{C}\left(\vec{A}\cdot\vec{B}\right). \tag{1.29}$$

Thus, the sedeonic product

$$\tilde{\mathbf{F}} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} = \mathbf{F}_0 + \vec{\mathbf{F}}$$

has the following components:

$$\begin{aligned} \mathbf{F}_0 &= \mathbf{A}_0\mathbf{B}_0 + \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3\,, \\ \mathbf{F}_1 &= \mathbf{A}_1\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_1 + i\mathbf{A}_2\mathbf{B}_3 - i\mathbf{A}_3\mathbf{B}_2\,, \\ \mathbf{F}_2 &= \mathbf{A}_2\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_2 + i\mathbf{A}_3\mathbf{B}_1 - i\mathbf{A}_1\mathbf{B}_3\,, \\ \mathbf{F}_3 &= \mathbf{A}_3\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_3 + i\mathbf{A}_1\mathbf{B}_2 - i\mathbf{A}_2\mathbf{B}_1\,. \end{aligned} \tag{1.30}$$

1.2. Spatial rotation and space-time conjugation

The rotation of the sedeon $\tilde{\mathbf{V}}$ on the angle θ around the absolute unit vector $\vec{n}$ is realized by sedeon

$$\tilde{\mathbf{U}} = \cos(\theta/2) + i\vec{n}\sin(\theta/2) \tag{1.31}$$

and by complex conjugated sedeon

$$\tilde{\mathbf{U}}^* = \cos(\theta/2) - i\vec{n}\sin(\theta/2)\,. \tag{1.32}$$

Note that these sedeons satisfy the following relation

$$\tilde{\mathbf{U}}^*\tilde{\mathbf{U}} = \tilde{\mathbf{U}}\tilde{\mathbf{U}}^* = 1\,. \tag{1.33}$$

The transformed sedeon $\tilde{\mathbf{V}}'$ is defined as the sedeonic product

$$\tilde{\mathbf{V}}' = \tilde{\mathbf{U}}^*\tilde{\mathbf{V}}\tilde{\mathbf{U}}\,. \tag{1.34}$$

Thus the transformed sedeon $\tilde{\mathbf{V}}'$ can be written in the following expanded form:

$$\begin{aligned} \tilde{\mathbf{V}}' &= \left[\cos(\theta/2) - i\vec{n}\sin(\theta/2)\right]\left(\mathbf{V}_0 + \vec{\mathbf{V}}\right)\left[\cos(\theta/2) + i\vec{n}\sin(\theta/2)\right] \\ &= \mathbf{V}_0 + \vec{\mathbf{V}}\cos\theta + (1-\cos\theta)\left(\vec{n}\cdot\vec{\mathbf{V}}\right)\vec{n} - i\sin\theta\left[\vec{n}\times\vec{\mathbf{V}}\right]. \end{aligned} \tag{1.35}$$

It is clearly seen that rotation does not transform the sedeon-scalar part, but sedeonic vector $\vec{\mathbf{V}}$ is rotated on the angle θ around $\vec{n}$.

The operations of time conjugation $(\hat{R}_{\mathbf{t}})$, space conjugation $(\hat{R}_{\mathbf{r}})$ and space-time conjugation $(\hat{R}_{\mathbf{tr}})$ are connected with transformations in $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ basis and can be presented as

$$\begin{aligned}
\hat{R}_{\mathbf{t}}\tilde{\mathbf{V}} &= \mathbf{e}_2\tilde{\mathbf{V}}\mathbf{e}_2 = \overline{\mathbf{V}}_0 - \mathbf{e}_1\overline{\mathbf{V}}_1 + \mathbf{e}_2\overline{\mathbf{V}}_2 - \mathbf{e}_3\overline{\mathbf{V}}_3, \\
\hat{R}_{\mathbf{r}}\tilde{\mathbf{V}} &= \mathbf{e}_1\tilde{\mathbf{V}}\mathbf{e}_1 = \overline{\mathbf{V}}_0 + \mathbf{e}_1\overline{\mathbf{V}}_1 - \mathbf{e}_2\overline{\mathbf{V}}_2 - \mathbf{e}_3\overline{\mathbf{V}}_3, \\
\hat{R}_{\mathbf{tr}}\tilde{\mathbf{V}} &= \mathbf{e}_3\tilde{\mathbf{V}}\mathbf{e}_3 = \overline{\mathbf{V}}_0 - \mathbf{e}_1\overline{\mathbf{V}}_1 - \mathbf{e}_2\overline{\mathbf{V}}_2 + \mathbf{e}_3\overline{\mathbf{V}}_3.
\end{aligned} \tag{1.36}$$

1.3. Subalgebras of smaller dimension

Sedeonic basis $\mathbf{e}_\alpha$, $\mathbf{a}_\beta$ allows one to construct different values of smaller dimension, which are differed in their properties with respect to the operations of spatial and time conjugation. For example, we can introduce the space-time double numbers as

$$D_{\mathbf{t}} = d_1 + \mathbf{e}_{\mathbf{t}} d_2, \tag{1.37}$$

$$D_{\mathbf{r}} = d_1 + \mathbf{e}_{\mathbf{r}} d_2, \tag{1.38}$$

$$D_{\mathbf{tr}} = d_1 + \mathbf{e}_{\mathbf{tr}} d_2, \tag{1.39}$$

where d_1 and d_2 are scalars. These values, on the one hand, have all properties of double numbers, but on the other hand, they are transformed differently by the space-time conjugation and sedeonic Lorentz transformations (see section 2.1).

We can also introduce the four-component values, which we call "quaterons" (in contrast to quaternions), in accordance with the following definitions:

$$\overset{\frown}{Q} = q_0\mathbf{a}_0 + \mathbf{e}_0\left(q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3\right), \tag{1.40}$$

$$\overset{\frown}{Q}_{\mathbf{t}} = q_0\mathbf{a}_0 + \mathbf{e}_{\mathbf{t}}\left(q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3\right), \tag{1.41}$$

$$\overset{\frown}{Q}_{\mathbf{r}} = q_0\mathbf{a}_0 + \mathbf{e}_{\mathbf{r}}\left(q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3\right), \tag{1.42}$$

$$\overset{\frown}{Q}_{\mathbf{tr}} = q_0\mathbf{a}_0 + \mathbf{e}_{\mathbf{tr}}\left(q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3\right). \tag{1.43}$$

The absolute quateron (1.40) is the sum of the absolute scalar and absolute vector. It is not changed under the operations of space-time conjugations. Time quateron $\breve{Q}_{\mathbf{t}}$, space quateron $\breve{Q}_{\mathbf{r}}$ and space-time quateron $\breve{Q}_{\mathbf{tr}}$ are transformed under the operations of space-time conjugation in accordance with the commutation rules for the basis elements $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$. For example, the time conjugation (see (1.36)) for quateron $\breve{Q}_{\mathbf{t}}$ is connected with the following transformation:

$$\hat{R}_{\mathbf{t}}\breve{Q}_{\mathbf{t}} = \mathbf{e_r}\breve{Q}_{\mathbf{t}}\mathbf{e_r} = q_0\mathbf{a_0} - \mathbf{e_t}\left(q_1\mathbf{a_1} + q_2\mathbf{a_2} + q_3\mathbf{a_3}\right). \qquad (1.41)$$

Furthermore, the sedeonic basis $\mathbf{e}_\alpha$, $\mathbf{a}_\beta$ also allows designing different types of space-time eight-component values named octons [20]:

$$\breve{G}_{\mathbf{t}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_t}G_{10} + \mathbf{e_t}\left(G_{11}\mathbf{a_1} + G_{12}\mathbf{a_2} + G_{13}\mathbf{a_3}\right), \qquad (1.42)$$

$$\breve{G}_{\mathbf{r}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_r}G_{20} + \mathbf{e_r}\left(G_{21}\mathbf{a_1} + G_{22}\mathbf{a_2} + G_{23}\mathbf{a_3}\right), \qquad (1.43)$$

$$\breve{G}_{\mathbf{tr}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_{tr}}G_{30} + \mathbf{e_{tr}}\left(G_{31}\mathbf{a_1} + G_{32}\mathbf{a_2} + G_{33}\mathbf{a_3}\right). \qquad (1.44)$$

Each of these subalgebras is closed with respect to the operation of Clifford multiplication (the ring). The application of spatial octons in electrodynamics and relativistic quantum mechanics was considered in [20-22].

1.4. Conclusion

The algebra of sedeons can be considered as a scalar-vector version of the Clifford algebra with specific rules of multiplication and commutation. The sedeonic basis elements $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are responsible for the spatial rotation, while the elements $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ are responsible for the space-time inversions. From the point of view of commutation and multiplication rules both these bases are equivalent.

In contrast to the Heaviside-Gibbs vector algebra the multiplication rules for vector basis in sedeonic algebra contain the imaginary unit (see Table 1). It enables the realization of scalar-vector algebra with Clifford product. Apparently, such possibility of vector basis multiplication was pointed first by A.Macfarlane [23]. Later the similar multiplication rules for matrix basis were applied by W.Pauli [24] and P.A.M.Dirac [25] in their spinor equations of quantum mechanics.

Chapter 2. Relativistic mechanics

2.1. Lorentz transformations

The relativistic event four-vector can be represented in the follow sedeonic form:

$$\tilde{\mathbf{S}} = i\mathbf{e}_{\mathbf{t}}ct + \mathbf{e}_{\mathbf{r}}\vec{r} \,, \tag{2.1}$$

where c is the speed of light, t is the absolute scalar of time and $\vec{r} = x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3$ is the absolute radius-vector. The sedeonic square of this value

$$\tilde{\mathbf{S}}\tilde{\mathbf{S}} = -c^2t^2 + x^2 + y^2 + z^2 \tag{2.2}$$

is the interval of event, which is the invariant of Lorentz transformation. In the frames of sedeonic algebra the transformation of values from one inertial coordinate system to another are carried out with the following sedeons:

$$\begin{aligned} \tilde{\mathbf{L}} &= \cosh\vartheta - \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta, \\ \tilde{\mathbf{L}}^* &= \cosh\vartheta + \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta, \end{aligned} \tag{2.3}$$

where $\tanh(2\vartheta) = v/c$; v is the velocity of uniform motion of the system along the absolute vector $\vec{m}$. Note, that

$$\tilde{\mathbf{L}}^*\tilde{\mathbf{L}} = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^* = 1 \,. \tag{2.4}$$

The transformed event four-vector $\tilde{\mathbf{S}}'$ is written as

$$\begin{aligned} \tilde{\mathbf{S}}' = \tilde{\mathbf{L}}^*\tilde{\mathbf{S}}\tilde{\mathbf{L}} &= \left(\cosh\vartheta + \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta\right)\left(i\mathbf{e}_{\mathbf{t}}ct + \mathbf{e}_{\mathbf{r}}\vec{r}\right)\left(\cosh\vartheta - \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta\right) \\ &= i\mathbf{e}_{\mathbf{t}}ct\cosh(2\vartheta) - i\mathbf{e}_{\mathbf{t}}\left(\vec{m}\cdot\vec{r}\right)\sinh(2\vartheta) + \mathbf{e}_{\mathbf{r}}\vec{r}\cosh 2\vartheta \\ &\quad - \mathbf{e}_{\mathbf{r}}ct\,\vec{m}\sinh(2\vartheta) + \mathbf{e}_{\mathbf{r}}\left(\vec{m}\cdot\vec{r}\right)\vec{m}\left(\cosh 2\vartheta - 1\right). \end{aligned} \tag{2.5}$$

Separating the values with $\mathbf{e}_1$ and $\mathbf{e}_2$ we get the well-known expressions for the time and coordinates transformations [26]:

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \,,\quad x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \,,\quad y' = y \,,\quad z' = z \,, \tag{2.6}$$

where x is the coordinate along the $\vec{m}$ unit vector.

Let us also consider the Lorentz transformation of the full sedeon $\tilde{\mathbf{V}}$. The transformed sedeon $\tilde{\mathbf{V}}'$ can be written as sedeonic product

$$\tilde{\mathbf{V}}' = \tilde{\mathbf{L}}^{*}\tilde{\mathbf{V}}\tilde{\mathbf{L}}\,. \tag{2.7}$$

In expanded form:

$$\begin{aligned}\tilde{\mathbf{V}}' &= \left(\cosh\vartheta + \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta\right)\left(\mathbf{V}_{\mathbf{0}} + \vec{\mathbf{V}}\right)\left(\cosh\vartheta - \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta\right) \\ &= \mathbf{V}_{\mathbf{0}}\cosh^{2}\vartheta - \mathbf{e}_{\mathbf{tr}}\mathbf{V}_{\mathbf{0}}\mathbf{e}_{\mathbf{tr}}\sinh^{2}\vartheta \\ &+ \left(\mathbf{e}_{\mathbf{tr}}\mathbf{V}_{\mathbf{0}} - \mathbf{V}_{\mathbf{0}}\mathbf{e}_{\mathbf{tr}}\right)\vec{m}\cosh\vartheta\sinh\vartheta + \vec{\mathbf{V}}\cosh^{2}\vartheta \\ &- \mathbf{e}_{\mathbf{tr}}\vec{m}\vec{\mathbf{V}}\vec{m}\mathbf{e}_{\mathbf{tr}}\sinh^{2}\vartheta + \left(\mathbf{e}_{\mathbf{tr}}\vec{m}\vec{\mathbf{V}} - \vec{\mathbf{V}}\vec{m}\mathbf{e}_{\mathbf{tr}}\right)\cosh\vartheta\sinh\vartheta\,.\end{aligned} \tag{2.8}$$

Rewriting the expression (2.8) with scalar (1.27) and vector (1.28) products, we get

$$\begin{aligned}\tilde{\mathbf{V}}' &= \mathbf{V}_{\mathbf{0}}\cosh^{2}\vartheta - \mathbf{e}_{\mathbf{tr}}\mathbf{V}_{\mathbf{0}}\mathbf{e}_{\mathbf{tr}}\sinh^{2}\vartheta \\ &+ \left(\mathbf{e}_{\mathbf{tr}}\mathbf{V}_{\mathbf{0}} - \mathbf{V}_{\mathbf{0}}\mathbf{e}_{\mathbf{tr}}\right)\vec{m}\cosh\vartheta\sinh\vartheta + \vec{\mathbf{V}}\cosh^{2}\vartheta \\ &- \mathbf{e}_{\mathbf{tr}}\vec{\mathbf{V}}\mathbf{e}_{\mathbf{tr}}\sinh^{2}\vartheta - 2\mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{\mathbf{V}}\right)\mathbf{e}_{\mathbf{tr}}\vec{m}\sinh^{2}\vartheta \\ &+ \left(\mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{\mathbf{V}}\right) - \left(\vec{\mathbf{V}}\cdot\vec{m}\right)\mathbf{e}_{\mathbf{tr}}\right)\cosh\vartheta\sinh\vartheta \\ &+ \left(\mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{\mathbf{V}}\right] - \left[\vec{\mathbf{V}}\times\vec{m}\right]\mathbf{e}_{\mathbf{tr}}\right)\cosh\vartheta\sinh\vartheta\,.\end{aligned} \tag{2.9}$$

Thus, the transformed sedeon have the following components:

$$\begin{aligned}& V' = V\,, \\ & V'_{\mathbf{tr}} = V_{\mathbf{tr}}\,, \\ & V'_{\mathbf{r}} = V_{\mathbf{r}}\cosh(2\vartheta) + \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\sinh(2\vartheta)\,, \\ & V'_{\mathbf{t}} = V_{\mathbf{t}}\cosh(2\vartheta) + \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\sinh(2\vartheta)\,, \\ & \vec{V}' = \vec{V}\cosh(2\vartheta) - \left(\vec{m}\cdot\vec{V}\right)\vec{m}\left(\cosh 2\vartheta - 1\right) \\ & \qquad + \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}_{\mathbf{tr}}\right]\sinh(2\vartheta)\,, \\ & \vec{V}'_{\mathbf{tr}} = \vec{V}_{\mathbf{tr}}\cosh(2\vartheta) - \left(\vec{m}\cdot\vec{V}_{\mathbf{tr}}\right)\vec{m}\left(\cosh 2\vartheta - 1\right) \\ & \qquad + \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}\right]\sinh(2\vartheta)\,, \\ & \vec{V}'_{\mathbf{r}} = \vec{V}_{\mathbf{r}} + \left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\vec{m}\left(\cosh 2\vartheta - 1\right) + \mathbf{e}_{\mathbf{tr}}V_{\mathbf{t}}\vec{m}\sinh(2\vartheta)\,, \\ & \vec{V}'_{\mathbf{t}} = \vec{V}_{\mathbf{t}} + \left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\vec{m}\left(\cosh 2\vartheta - 1\right) + \mathbf{e}_{\mathbf{tr}}V_{\mathbf{r}}\vec{m}\sinh(2\vartheta)\,.\end{aligned} \tag{2.10}$$

2.2. Relativistic momentum and angular momentum

In relativistic physics an important value is the four-vector of energy-momentum of the relativistic particle. In sedeonic algebra it can be represented as

$$\tilde{\mathbf{E}} = i\mathbf{e}_{\mathbf{t}}E + \mathbf{e}_{\mathbf{r}}c\vec{p} \,, \qquad (2.11)$$

where E is the energy and $\vec{p}$ is the momentum of particle. The square of this value

$$\left(i\mathbf{e}_{\mathbf{t}}E + \mathbf{e}_{\mathbf{r}}c\vec{p}\right)\left(i\mathbf{e}_{\mathbf{t}}E + \mathbf{e}_{\mathbf{r}}c\vec{p}\right) = -E^2 + c^2\vec{p}^{\,2} \qquad (2.12)$$

is the invariant of Lorentz transformations and connected with particle inertial mass m_0 by Einstein relation

$$E^2 - c^2\vec{p}^{\,2} - m_0^2c^4 = 0 \,. \qquad (2.13)$$

In sedeonic algebra this expression can be presented as the product of two the same factors

$$\left(i\mathbf{e}_{\mathbf{t}}E + \mathbf{e}_{\mathbf{r}}c\vec{p} + \mathbf{e}_{\mathbf{tr}}m_0c^2\right)\left(i\mathbf{e}_{\mathbf{t}}E + \mathbf{e}_{\mathbf{r}}c\vec{p} + \mathbf{e}_{\mathbf{tr}}m_0c^2\right) = 0 \,, \qquad (2.14)$$

that will be used further for constructing of quantum mechanics and field theory equations.

The generalized angular momentum for relativistic particle can be written as follows:

$$\tilde{\mathbf{M}} = \frac{1}{c}\tilde{\mathbf{E}}\tilde{\mathbf{S}} = \frac{1}{c}\left(i\mathbf{e}_{\mathbf{t}}E + \mathbf{e}_{\mathbf{r}}c\vec{p}\right)\left(i\mathbf{e}_{\mathbf{t}}ct + \mathbf{e}_{\mathbf{r}}\vec{r}\right) . \qquad (2.15)$$

Performing sedeonic multiplication we get

$$\tilde{\mathbf{M}} = -Et + \left(\vec{p}\cdot\vec{r}\right) + \left[\vec{p}\times\vec{r}\right] + \mathbf{e}_{\mathbf{tr}}c\vec{p}t - \mathbf{e}_{\mathbf{tr}}\frac{1}{c}E\vec{r} \,. \qquad (2.16)$$

Chapter 3. Quantum mechanics and field theory

3.1. Generalized sedeonic wave equation

The wave function of the free particle should satisfy an equation, which is obtained from the Einstein relation between particle energy and momentum

$$E^2 - c^2 p^2 - m_0^2 c^4 = 0 \tag{3.1}$$

by means of changing classical energy E and momentum $\vec{p}$ on corresponding quantum mechanical operators [27]:

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \quad \text{and} \quad \hat{\vec{p}} = -i\hbar\vec{\nabla}\ , \tag{3.2}$$

where $\hbar$ is Planck constant and $\vec{\nabla}$ is the gradient operator, which has the following form:

$$\vec{\nabla} = \mathbf{a_1}\frac{\partial}{\partial x} + \mathbf{a_2}\frac{\partial}{\partial y} + \mathbf{a_3}\frac{\partial}{\partial z}\ . \tag{3.3}$$

In sedeonic algebra the Einstein relation (3.1) for operators (3.2) can be written as

$$\left(i\mathbf{e_t}\hat{E} + \mathbf{e_r}c\hat{\vec{p}} + \mathbf{e_{tr}}m_0c^2\right)\left(i\mathbf{e_t}\hat{E} + \mathbf{e_r}c\hat{\vec{p}} + \mathbf{e_{tr}}m_0c^2\right) = 0\ . \tag{3.4}$$

Let us consider the wave function in the form of space-time sedeon

$$\tilde{\boldsymbol{\psi}}(\vec{r},t) = \boldsymbol{\psi}_\mathbf{0}(\vec{r},t) + \vec{\boldsymbol{\psi}}(\vec{r},t)\ , \tag{3.5}$$

then the generalized sedeonic wave equation, corresponding to the operator equation (3.4), is written in the following symmetric form:

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}\frac{m_0c}{\hbar}\right)\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}\frac{m_0c}{\hbar}\right)\tilde{\boldsymbol{\psi}} = 0\ . \tag{3.6}$$

In this equation the parameter m_0 is the rest mass of particle.

Besides, there is a special class of particles, which is described by the first-order wave equation [27]:

$$\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_{\mathbf{r}}\vec{\nabla}-i\mathbf{e}_{\mathbf{tr}}\frac{m_0 c}{\hbar}\right)\tilde{\boldsymbol{\psi}}=0\,. \tag{3.7}$$

Obviously, that for such particles the equation (3.6) is satisfied automatically.

The sedeonic quantum equation (3.6) admits the field interpretation. To simplify the further presentation we introduce the following operators:

$$\begin{aligned}\partial_{\mathbf{t}} &= \mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}\,,\\ \vec{\nabla}_{\mathbf{r}} &= \mathbf{e}_{\mathbf{r}}\vec{\nabla}\,,\\ m_{\mathbf{tr}} &= \mathbf{e}_{\mathbf{tr}}\frac{m_0 c}{\hbar}\,,\end{aligned} \tag{3.8}$$

then the equation (3.6) takes the form

$$\left(i\partial_{\mathbf{t}}-\vec{\nabla}_{\mathbf{r}}-im_{\mathbf{tr}}\right)\left(i\partial_{\mathbf{t}}-\vec{\nabla}_{\mathbf{r}}-im_{\mathbf{tr}}\right)\tilde{\boldsymbol{\psi}}=0\,. \tag{3.9}$$

Let us consider sequential action of the operators on the left side of (3.9). After the action of the first operator we obtain

$$\begin{aligned}&\left(i\partial_{\mathbf{t}}-\vec{\nabla}_{\mathbf{r}}-im_{\mathbf{tr}}\right)\tilde{\boldsymbol{\psi}}=i\partial_{\mathbf{t}}\boldsymbol{\psi}_{\mathbf{0}}+i\partial_{\mathbf{t}}\vec{\boldsymbol{\psi}}-\vec{\nabla}_{\mathbf{r}}\boldsymbol{\psi}_{\mathbf{0}}\\ &-\left(\vec{\nabla}_{\mathbf{r}}\cdot\vec{\boldsymbol{\psi}}\right)-\left[\vec{\nabla}_{\mathbf{r}}\times\vec{\boldsymbol{\psi}}\right]-im_{\mathbf{tr}}\boldsymbol{\psi}_{\mathbf{0}}-im_{\mathbf{tr}}\vec{\boldsymbol{\psi}}.\end{aligned} \tag{3.10}$$

Introducing scalar and vector field strengths according

$$\mathbf{E}_{\mathbf{0}}=i\partial_{\mathbf{t}}\boldsymbol{\psi}_{\mathbf{0}}-\left(\vec{\nabla}_{\mathbf{r}}\cdot\vec{\boldsymbol{\psi}}\right)-im_{\mathbf{tr}}\boldsymbol{\psi}_{\mathbf{0}}\,, \tag{3.11}$$

$$\vec{\mathbf{E}}=i\partial_{\mathbf{t}}\vec{\boldsymbol{\psi}}-\vec{\nabla}_{\mathbf{r}}\boldsymbol{\psi}_{\mathbf{0}}-\left[\vec{\nabla}_{\mathbf{r}}\times\vec{\boldsymbol{\psi}}\right]-im_{\mathbf{tr}}\vec{\boldsymbol{\psi}}\,, \tag{3.12}$$

expression (3.10) can be rewritten as

$$\left(i\partial_{\mathbf{t}}-\vec{\nabla}_{\mathbf{r}}-im_{\mathbf{tr}}\right)\tilde{\boldsymbol{\psi}}=\mathbf{E}_{\mathbf{0}}+\vec{\mathbf{E}}\,. \tag{3.13}$$

Then the wave equation (3.9) takes the form

$$\left(i\partial_{\mathbf{t}}-\vec{\nabla}_{\mathbf{r}}-im_{\mathbf{tr}}\right)\left(\mathbf{E}_{\mathbf{0}}+\vec{\mathbf{E}}\right)=0\,. \tag{3.14}$$

Producing the action of the operator in (3.14) and separating sedeon-scalar and sedeon-vector parts we obtain a system of first-order equations similar to the Maxwell's equations:

$$i\partial_{\mathbf{t}}\mathbf{E_0} - \left(\vec{\nabla}_{\mathbf{r}} \cdot \vec{\mathbf{E}}\right) - im_{\mathbf{tr}}\mathbf{E_0} = 0,$$
$$i\partial_{\mathbf{t}}\vec{\mathbf{E}} - \left[\vec{\nabla}_{\mathbf{r}} \times \vec{\mathbf{E}}\right] - im_{\mathbf{tr}}\vec{\mathbf{E}} - \vec{\nabla}_{\mathbf{r}}\mathbf{E_0} = 0. \tag{3.15}$$

In fact, the first- and second-order wave equation of describe quantum fields that carry information on the kinematic properties of quantum particles. The dispersion characteristic of these wave fields coincide with the Einstein relation for the energy and momentum of a particle. Note, that the first-order equation (3.7) describes the special case of quantum fields with field strengths equal to zero. More detailed the quantum mechanics of relativistic particles will be discussed in the Chapter 7.

The generalized sedeonic wave equation (3.6) has another interpretation as the wave equation for the force massive fields [16]. In this case, the field sources are appropriate charges and currents, so that in addition to the homogeneous equation, which describes the free field, we have non-homogeneous equation

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\tilde{\mathbf{W}} = \tilde{\mathbf{J}}, \tag{3.16}$$

where by $\tilde{\mathbf{J}}$ we have identified the scalar-vector source of field. In this case, the wave function has the meaning of the field potential and the parameter m_0 is the mass of the quantum of field. Of course, in the case of zero m_0 the equation (3.16) should describe electromagnetic and weak gravitational fields. The sedeonic theory of massive and massless force fields will be discussed in detail in the subsequent sections.

Chapter 4. Electromagnetic field

4.1. Sedeonic form of electromagnetic field equations

The sedeonic wave equation for the electromagnetic field in a vacuum is written as follows

$$\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\tilde{\mathbf{W}}=\tilde{\mathbf{J}}\,. \tag{4.1}$$

The potential of the electromagnetic field has the form:

$$\tilde{\mathbf{W}}=\ i\mathbf{e}_{\mathbf{t}}\varphi_e+\mathbf{e}_{\mathbf{r}}\vec{A}_e\,, \tag{4.2}$$

where φ_e is scalar potential (time component), $\vec{A}_e$ is vector potential (space component). A source of the field is written as follows:

$$\tilde{\mathbf{J}}=-4\pi i\mathbf{e}_{\mathbf{t}}\rho_e-\mathbf{e}_{\mathbf{r}}\frac{4\pi}{c}\vec{j}_e \tag{4.3}$$

where ρ_e is a volume density of electric charge, $\vec{j}_e$ is a volume density of electric current.

The equation (4.1) is a compact universal relation, which can be represented either as a system of wave equations for the potentials of the field or in the form of Maxwell's equations for the field strengths. Indeed, producing the multiplication of operators in equation (4.1) and separating the scalar and vector parts, we obtain a system of wave equations:

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\varphi_e=4\pi\rho_e\,, \tag{4.4}$$

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{A}_e=\frac{4\pi}{c}\vec{j}_e\,. \tag{4.5}$$

Here we assume that the potentials are described by twice differentiable functions, so that $\left[\vec{\nabla}\times\vec{\nabla}\right]\tilde{\mathbf{W}}=0$. On the other hand, performing the step-by-step action of operators in equation (4.1), we have first

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(i\mathbf{e_t}\varphi_e+\mathbf{e_r}\vec{A}_e\right)$$
$$=-\frac{1}{c}\frac{\partial\varphi_e}{\partial t}-\mathbf{e_{tr}}\frac{1}{c}\frac{\partial\vec{A}_e}{\partial t}-\mathbf{e_{tr}}\vec{\nabla}\varphi_e-\left(\vec{\nabla}\cdot\vec{A}_e\right)-\left[\vec{\nabla}\times\vec{A}_e\right]. \tag{4.6}$$

Let us introduce the scalar and vector strengths of electromagnetic field:

$$f_e=\frac{1}{c}\frac{\partial\varphi_e}{\partial t}+\left(\vec{\nabla}\cdot\vec{A}_e\right),$$
$$\vec{E}_e=-\vec{\nabla}\varphi_e-\frac{1}{c}\frac{\partial\vec{A}_e}{\partial t}, \tag{4.7}$$
$$\vec{H}_e=-i\left[\vec{\nabla}\times\vec{A}_e\right].$$

Then the expression (4.6) can be represented as

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(i\mathbf{e_t}\varphi_e+\mathbf{e_r}\vec{A}_e\right)=-f_e+\mathbf{e_{tr}}\vec{E}_e-i\vec{H}_e\,, \tag{4.8}$$

and equation (4.1) can be rewritten in the following form:

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(-f_e+\mathbf{e_{tr}}\vec{E}_e-i\vec{H}_e\right)=-\,4\pi i\mathbf{e_t}\rho_e-\mathbf{e_r}\frac{4\pi}{c}\vec{j}_e. \tag{4.9}$$

Applying the operator

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)$$

to both parts of equation (4.9) and separating values with different space-time properties we obtain the wave equations for the strengths of electromagnetic field:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)f_e=\frac{4\pi}{c}\left\{\frac{\partial\rho_e}{\partial t}+\left(\vec{\nabla}\cdot\vec{j}_e\right)\right\}, \tag{4.10}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{E}_e=-4\pi\vec{\nabla}\rho_e-\frac{4\pi}{c^2}\frac{\partial\vec{j}_e}{\partial t}, \tag{4.11}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{H}_e=-i\frac{4\pi}{c}\left[\vec{\nabla}\times\vec{j}_e\right]\quad. \tag{4.12}$$

Assuming the electrical charge conservation

$$\frac{\partial \rho_e}{\partial t} + \left(\vec{\nabla} \cdot \vec{j}_e\right) = 0\,, \tag{4.13}$$

we note that equation (4.10) has no sources and the scalar field f_e can be chosen equal to zero. This is equivalent to the Lorentz gauge condition:

$$f_e = \frac{1}{c}\frac{\partial \varphi_e}{\partial t} + (\vec{\nabla} \cdot \vec{A}_e) = 0\ . \tag{4.14}$$

In the Lorentz gauge the equation (4.9) takes the form

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla}\right)\left(\mathbf{e_{tr}}\vec{E}_e - i\vec{H}_e\right) = -\,4\pi i\mathbf{e_t}\rho_e - \mathbf{e_r}\frac{4\pi}{c}\vec{j}_e. \tag{4.15}$$

Producing action of the operator on the left side of (4.15), we have the following sedeonic equation:

$$\begin{aligned}
&\mathbf{e_r}\frac{1}{c}\frac{\partial \vec{E}_e}{\partial t} - i\mathbf{e_t}\left(\vec{\nabla} \cdot \vec{E}_e\right) - i\mathbf{e_t}\left[\vec{\nabla} \times \vec{E}_e\right] \\
&+ \mathbf{e_t}\frac{1}{c}\frac{\partial \vec{H}_e}{\partial t} + i\mathbf{e_r}\left(\vec{\nabla} \cdot \vec{H}_e\right) + i\mathbf{e_r}\left[\vec{\nabla} \times \vec{H}_e\right] \\
&= -\,4\pi i\mathbf{e_t}\rho_e - \mathbf{e_r}\frac{4\pi}{c}\vec{j}_e\,.
\end{aligned} \tag{4.16}$$

Separating in (4.16) the values with different properties, we obtain a system of first-order equations

$$\begin{aligned}
&-i\left[\vec{\nabla} \times \vec{E}_e\right] = -\frac{1}{c}\frac{\partial \vec{H}_e}{\partial t}, \\
&-i\left[\vec{\nabla} \times \vec{H}_e\right] = \frac{4\pi}{c}\vec{j}_e + \frac{1}{c}\frac{\partial \vec{E}_e}{\partial t}, \\
&\left(\vec{\nabla} \cdot \vec{E}_e\right) = 4\pi\rho_e, \\
&\left(\vec{\nabla} \cdot \vec{H}_e\right) = 0,
\end{aligned} \tag{4.17}$$

which coincides with the Maxwell equations.

4.2. Energy and momentum of electromagnetic field

The sedeonic algebra allows one to provide the combined calculus with the values of different type. In this section we consider the relations for the energy and momentum of electromagnetic field.

Multiplying both parts of equation (4.15) on the sedeon ($\mathbf{e}_{\mathbf{tr}}\vec{E}_e - i\vec{H}_e$), from the left we obtain

$$\begin{aligned}
&-i\mathbf{e}_{\mathbf{t}}\left\{\frac{1}{2c}\frac{\partial}{\partial t}\left\{\left(\vec{E}_e\right)^2+\left(\vec{H}_e\right)^2\right\}-i\left(\vec{\nabla}\cdot\left[\vec{E}_e\times\vec{H}_e\right]\right)\right\} \\
&+\mathbf{e}_{\mathbf{r}}\left\{i\frac{1}{c}\left(\vec{E}_e\cdot\frac{\partial\vec{H}_e}{\partial t}\right)-i\frac{1}{c}\left(\vec{H}_e\cdot\frac{\partial\vec{E}_e}{\partial t}\right)+\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)+\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)\right\} \\
&+\mathbf{e}_{\mathbf{t}}\left\{-i\frac{1}{c}\left[\vec{E}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]-i\frac{1}{c}\left[\vec{H}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]\right. \\
&\left.+\vec{E}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)-\vec{H}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)+\left[\vec{E}_e\times\left[\vec{\nabla}\times\vec{H}_e\right]\right]-\left[\vec{H}_e\times\left[\vec{\nabla}\times\vec{E}_e\right]\right]\right\} \qquad (4.18) \\
&+\mathbf{e}_{\mathbf{r}}\left\{i\frac{1}{c}\frac{\partial}{\partial t}\left[\vec{E}_e\times\vec{H}_e\right]-\frac{1}{2}\vec{\nabla}\left\{\left(\vec{E}_e\right)^2+\left(\vec{H}_e\right)^2\right\}+\left(\vec{\nabla}\cdot\vec{E}_e\right)\vec{E}_e+\left(\vec{\nabla}\cdot\vec{H}_e\right)\vec{H}_e\right\} \\
&=i\mathbf{e}_{\mathbf{t}}\frac{4\pi}{c}\left(\vec{E}_e\cdot\vec{j}_e\right)+i\mathbf{e}_{\mathbf{r}}\frac{4\pi}{c}\left(\vec{H}_e\cdot\vec{j}_e\right)-4\pi\mathbf{e}_{\mathbf{t}}\left\{\rho_e\vec{H}_e-i\frac{1}{c}\left[\vec{E}_e\times\vec{j}_e\right]\right\} \\
&+4\pi\mathbf{e}_{\mathbf{r}}\left\{\rho_e\vec{E}_e+i\frac{1}{c}\left[\vec{H}_e\times\vec{j}_e\right]\right\}.
\end{aligned}$$

Note that in this expression and further the operator applies to all right expression. For example, for any two vectors $\vec{A}$ and $\vec{B}$ we have:

$$\left(\vec{\nabla}\cdot\vec{A}\right)\vec{B}=\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)+\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}\,. \qquad (4.19)$$

Equating in (4.18) the components with different space-time properties we get

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{E}_e^2+\vec{H}_e^2\right)-i\frac{c}{4\pi}\left(\vec{\nabla}\cdot\left[\vec{E}_e\times\vec{H}_e\right]\right)+\left(\vec{E}_e\cdot\vec{j}_e\right)=0\,, \qquad (4.20)$$

$$\begin{aligned}
&\frac{1}{8\pi}\vec{\nabla}\left(\vec{E}_e^2+\vec{H}_e^2\right)-i\frac{1}{4\pi c}\frac{\partial}{\partial t}\left[\vec{E}_e\times\vec{H}_e\right] \\
&-\frac{1}{4\pi}\left\{\left(\vec{\nabla}\cdot\vec{E}_e\right)\vec{E}_e+\left(\vec{\nabla}\cdot\vec{H}_e\right)\vec{H}_e\right\}+\rho_e\vec{E}_e+i\left[\vec{H}_e\times\vec{j}_e\right]=0,
\end{aligned} \qquad (4.21)$$

$$\frac{1}{4\pi}\left\{\left(\vec{E}_e\cdot\frac{\partial\vec{H}_e}{\partial t}\right)-\left(\vec{H}_e\cdot\frac{\partial\vec{E}_e}{\partial t}\right)\right\}$$
$$-i\frac{c}{4\pi}\left\{\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)+\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)\right\}-\left(\vec{H}_e\cdot\vec{j}_e\right)=0, \tag{4.22}$$

$$-i\frac{1}{4\pi}\left\{\left[\vec{E}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]+\left[\vec{H}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]\right\}+\frac{c}{4\pi}\left\{\vec{E}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)-\vec{H}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)\right\}$$
$$+\frac{c}{4\pi}\left\{\left[\vec{E}_e\times\left[\vec{\nabla}\times\vec{H}_e\right]\right]-\left[\vec{H}_e\times\left[\vec{\nabla}\times\vec{E}_e\right]\right]\right\}+c\vec{H}_e\rho_e-i\left[\vec{E}_e\times\vec{j}_e\right]=0. \tag{4.23}$$

The expression (4.20) is the very well known relation named as Pointing theorem. The value

$$w=\frac{\vec{E}_e^2+\vec{H}_e^2}{8\pi} \tag{4.24}$$

is the volume density of field energy, while vector

$$\vec{P}=-i\frac{c}{4\pi}\left[\vec{E}_e\times\vec{H}_e\right] \tag{4.25}$$

is the energy flux density vector (Pointing's vector).

4.3. Relations for Lorentz invariants of electromagnetic field

Using sedeonic algebra it is easy to derive the relations for the values

$$I_1=\vec{E}_e^2-\vec{H}_e^2,$$
$$I_2=\left(\vec{E}_e\cdot\vec{H}_e\right), \tag{4.26}$$

which are Lorentz invariants of electromagnetic field. Multiplying both parts of equation (4.15) on sedeon ($\mathbf{e_{tr}}\vec{E}_e+i\vec{H}_e$) from the left we have:

$$-i\mathbf{e_t}\left\{\frac{1}{c}\left(\vec{E}_e\cdot\frac{\partial\vec{E}_e}{\partial t}\right)-\frac{1}{c}\left(\vec{H}_e\cdot\frac{\partial\vec{H}_e}{\partial t}\right)+i\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)+i\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)\right\}$$

$$+i\mathbf{e_r}\left\{\frac{1}{c}\left(\vec{E}_e\cdot\frac{\partial\vec{H}_e}{\partial t}\right)+\frac{1}{c}\left(\vec{H}_e\cdot\frac{\partial\vec{E}_e}{\partial t}\right)-i\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)+i\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)\right\}$$

$$+\mathbf{e_t}\left\{-i\frac{1}{c}\left[\vec{E}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]+i\frac{1}{c}\left[\vec{H}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]\right.$$

$$\left.+\vec{E}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)+\vec{H}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)+\left[\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right]+\left[\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right]\right\} \qquad (4.27)$$

$$+\mathbf{e_r}\left\{i\frac{1}{c}\left[\vec{E}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]+i\frac{1}{c}\left[\vec{H}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]+\vec{E}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)-\vec{H}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)\right.$$

$$\left.+\left[\vec{E}_e\times\left[\vec{\nabla}\times\vec{E}_e\right]\right]-\left[\vec{H}_e\times\left[\vec{\nabla}\times\vec{H}_e\right]\right]\right\}$$

$$=i\mathbf{e_t}\frac{4\pi}{c}\left(\vec{E}_e\cdot\vec{j}_e\right)-i\mathbf{e_r}\frac{4\pi}{c}\left(\vec{H}_e\cdot\vec{j}_e\right)+4\pi\mathbf{e_t}\left\{\rho_e\vec{H}_e+i\frac{1}{c}\left[\vec{E}_e\times\vec{j}_e\right]\right\}$$

$$+4\pi\mathbf{e_r}\left\{\rho_e\vec{E}_e-i\frac{1}{c}\left[\vec{H}_e\times\vec{j}_e\right]\right\}.$$

Separating the values of different types, we obtain the following relations for the Lorentz invariants of the electromagnetic field:

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left\{\vec{E}_e^2-\vec{H}_e^2\right\}$$
$$=-\left(\vec{E}_e\cdot\vec{j}_e\right)-i\frac{c}{4\pi}\left\{\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)+\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)\right\}, \qquad (4.28)$$

$$\frac{c}{8\pi}\vec{\nabla}\left\{\vec{E}_e^2-\vec{H}_e^2\right\}$$
$$=\frac{c}{4\pi}\left\{\vec{E}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)+\left(\vec{E}_e\cdot\vec{\nabla}\right)\vec{E}_e-\vec{H}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)-\left(\vec{H}_e\cdot\vec{\nabla}\right)\vec{H}_e\right\} \qquad (4.29)$$
$$+i\frac{1}{4\pi}\left\{\left[\vec{E}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]+\left[\vec{H}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]\right\}-c\rho_e\vec{E}_e+i\left[\vec{H}_e\times\vec{j}_e\right],$$

$$\frac{1}{4\pi}\frac{\partial}{\partial t}\left\{\left(\vec{E}_e\cdot\vec{H}_e\right)\right\}$$
$$=i\frac{c}{4\pi}\left\{\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)-\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)\right\}-\left(\vec{H}_e\cdot\vec{j}_e\right), \qquad (4.30)$$

$$\frac{c}{4\pi}\vec{\nabla}\left\{\left(\vec{E}_e\cdot\vec{H}_e\right)\right\}$$

$$=\frac{c}{4\pi}\left\{\vec{E}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)+\vec{H}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)-\left(\vec{E}_e\cdot\vec{\nabla}\right)\vec{H}_{g}+\left(\vec{H}_e\cdot\vec{\nabla}\right)\vec{E}_e\right\} \qquad (4.31)$$

$$-i\frac{1}{4\pi}\left\{\left[\vec{E}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]-\left[\vec{H}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]\right\}-c\vec{H}_e\rho_e+i\left[\vec{E}_e\times\vec{j}_e\right].$$

4.4. Supersymmetric equations of electromagnetic field

The question of symmetry between electric and magnetic charges was considered first by P.A.M.Dirac [28, 29]. Taking into account the hypothetical magnetic charges (magnetic monopoles) and corresponding current the system of Maxwell equations looks absolutely symmetric [30]. In terms of sedeonic algebra the symmetric wave equation for the electromagnetic field can be written as

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\tilde{\mathbf{W}}=\tilde{\mathbf{J}}\ . \qquad (4.32)$$

Here $\tilde{\mathbf{W}}$ is sedeon potential

$$\tilde{\mathbf{W}}=i\mathbf{e}_1\varphi_e-i\mathbf{e}_2\varphi_m+\mathbf{e}_1\vec{A}_m+\mathbf{e}_2\vec{A}_e, \qquad (4.33)$$

where φ_e is electric scalar potential, φ_m is magnetic scalar potential, $\vec{A}_e$ is electric vector potential, $\vec{A}_m$ is magnetic vector potential. The sedeonic source is

$$\tilde{\mathbf{J}}=-i\mathbf{e}_1 4\pi\rho_e-\mathbf{e}_2\frac{4\pi}{c}\vec{j}_e+i\mathbf{e}_2 4\pi\rho_m-\mathbf{e}_1\frac{4\pi}{c}\vec{j}_m, \qquad (4.34)$$

where ρ_m is volume density of magnetic charge, $\vec{j}_m$ is density of magnetic current.

Equation (4.32) is equivalent to the eight scalar equations for the components of the potentials. Separating in (4.32) space-time and the scalar-vector parts we obtain the following wave equations for the potentials:

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\varphi_e = 4\pi\rho_e\,, \tag{4.35}$$

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{A}_e = \frac{4\pi}{c}\vec{j}_e\,, \tag{4.36}$$

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\varphi_m = 4\pi\rho_m\,, \tag{4.37}$$

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{A}_m = \frac{4\pi}{c}\vec{j}_m\,. \tag{4.38}$$

Introducing the scalar and vector field strength, according to the following definitions:

$$\begin{aligned}
e &= \frac{1}{c}\frac{\partial\varphi_e}{\partial t}+\left(\vec{\nabla}\cdot\vec{A}_e\right),\\
h &= \frac{1}{c}\frac{\partial\varphi_m}{\partial t}+\left(\vec{\nabla}\cdot\vec{A}_m\right),\\
\vec{E} &= -\frac{1}{c}\frac{\partial\vec{A}_e}{\partial t}-\vec{\nabla}\varphi_e+i\left[\vec{\nabla}\times\vec{A}_m\right],\\
\vec{H} &= -\frac{1}{c}\frac{\partial\vec{A}_m}{\partial t}-\vec{\nabla}\varphi_m-i\left[\vec{\nabla}\times\vec{A}_e\right],
\end{aligned} \tag{4.39}$$

we obtain

$$\begin{aligned}
&\left(i\mathbf{e_1}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_2}\vec{\nabla}\right)\left(i\mathbf{e_1}\varphi_e-i\mathbf{e_2}\varphi_m+\mathbf{e_1}\vec{A}_m+\mathbf{e_2}\vec{A}_e\right)\\
&=-e+i\mathbf{e_3}h-i\vec{H}+\mathbf{e_3}\vec{E},
\end{aligned} \tag{4.40}$$

and the wave equation (4.32) is reduced to

$$\begin{aligned}
&\left(i\mathbf{e_1}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_2}\vec{\nabla}\right)\left(-e+i\mathbf{e_3}h-i\vec{H}+\mathbf{e_3}\vec{E}\right)\\
&=-i\mathbf{e_1}4\pi\rho_e-\mathbf{e_2}\frac{4\pi}{c}\vec{j}_e+i\mathbf{e_2}4\pi\rho_m-\mathbf{e_1}\frac{4\pi}{c}\vec{j}_m.
\end{aligned} \tag{4.41}$$

Producing action of the operator on the left side of this equation and separating the values with different space-time properties, we obtain the system of equations for the fields, similar to the system of Maxwell's equations in electrodynamics:

$$\begin{aligned}
&\frac{1}{c}\frac{\partial e}{\partial t}+\left(\vec{\nabla}\cdot\vec{E}\right)=4\pi\rho_e,\\
&\frac{1}{c}\frac{\partial h}{\partial t}+\left(\vec{\nabla}\cdot\vec{H}\right)=4\pi\rho_m,\\
&\frac{1}{c}\frac{\partial \vec{E}}{\partial t}+\vec{\nabla}e+i\left[\vec{\nabla}\times\vec{H}\right]=-\frac{4\pi}{c}\vec{j}_e,\\
&\frac{1}{c}\frac{\partial \vec{H}}{\partial t}+\vec{\nabla}h-i\left[\vec{\nabla}\times\vec{E}\right]=-\frac{4\pi}{c}\vec{j}_m.
\end{aligned} \tag{4.42}$$

Equations (4.42) form a closed system of eight scalar equations for the eight components of the electromagnetic field.

Scalar and vector field strengths (4.39) and equations (4.42) have the gauge invariance with respect to the gradient transformations of the following form:

$$\begin{aligned}
&\varphi_e \Rightarrow \varphi_e+\frac{\partial\varepsilon_e}{\partial t},\\
&\vec{A}_e \Rightarrow \vec{A}_e+\vec{\nabla}\varepsilon_e,\\
&\varphi_m \Rightarrow \varphi_m+\frac{\partial\varepsilon_m}{\partial t},\\
&\vec{A}_m \Rightarrow \vec{A}_m+\vec{\nabla}\varepsilon_m,
\end{aligned} \tag{4.43}$$

where ε_e and ε_m are arbitrary scalar functions of coordinates and time, satisfying the homogeneous wave equation. Indeed it is easy to verify that the transformations (4.43) do not change the field strengths (4.39) and, consequently, the equations (4.42).

Applying the operator

$$\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)$$

to both sides of (4.41) and separating the values with different s space-time properties, we obtain the wave equations for the field strengths:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)e=\frac{4\pi}{c}\left\{\frac{\partial\rho_e}{\partial t}+\left(\vec{\nabla}\cdot\vec{j}_e\right)\right\}, \tag{4.44}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)h=\frac{4\pi}{c}\left\{\frac{\partial\rho_m}{\partial t}+\left(\vec{\nabla}\cdot\vec{j}_m\right)\right\}, \tag{4.45}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{E}=-4\pi\vec{\nabla}\rho_e-\frac{4\pi}{c^2}\frac{\partial\vec{j}_e}{\partial t}+i\frac{4\pi}{c}\left[\vec{\nabla}\times\vec{j}_m\right], \tag{4.46}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{H}=-4\pi\vec{\nabla}\rho_m-\frac{4\pi}{c^2}\frac{\partial\vec{j}_m}{\partial t}-i\frac{4\pi}{c}\left[\vec{\nabla}\times\vec{j}_e\right]. \tag{4.47}$$

If in the physical system there is conservation of electric and magnetic charges

$$\frac{\partial\rho_e}{\partial t}+\left(\vec{\nabla}\cdot\vec{j}_e\right)=0, \tag{4.48}$$

$$\frac{\partial\rho_m}{\partial t}+\left(\vec{\nabla}\cdot\vec{j}_m\right)=0, \tag{4.49}$$

then the equations (4.44) and (4.45) do not have sources, and one can choose the scalar fields e and h equal to zero. This is equivalent to the Lorentz gauge conditions for the electric and magnetic potentials:

$$e=\frac{1}{c}\frac{\partial\varphi_e}{\partial t}+(\vec{\nabla}\cdot\vec{A}_e)=0, \tag{4.50}$$

$$h=\frac{1}{c}\frac{\partial\varphi_m}{\partial t}+(\vec{\nabla}\cdot\vec{A}_m)=0. \tag{4.51}$$

In the Lorentz gauge the Maxwell equations (4.42) can be written as follows:

$$\left(\vec{\nabla}\cdot\vec{E}\right)=4\pi\rho_e, \tag{4.52}$$

$$\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+i\left[\vec{\nabla}\times\vec{H}\right]=-\frac{4\pi}{c}\vec{j}_e, \tag{4.53}$$

$$\left(\vec{\nabla}\cdot\vec{H}\right)=4\pi\rho_m, \tag{4.54}$$

$$\frac{1}{c}\frac{\partial\vec{H}}{\partial t}-i\left[\vec{\nabla}\times\vec{E}\right]=-\frac{4\pi}{c}\vec{j}_m. \tag{4.55}$$

As established experimentally in our part of the universe magnetic charges and currents are not observed, and then for the description of the phenomena in the Earth's environment must be put

$$\rho_m = 0, \quad \vec{j}_m = 0. \tag{4.56}$$

This leads us to the fact that the equations (4.37) and (4.38) do not have sources and we can choose magnetic potentials φ_m and $\vec{A}_m$ equal to zero. In addition, the magnetic sources disappear on the right sides of equations (4.54) and (4.55). Thus, in the particular case of the absence of magnetic charges and the electric charge conservation, we come to the standard system of Maxwell's equations:

$$\begin{aligned} &\left(\vec{\nabla}\cdot\vec{E}\right) = 4\pi\rho_e, \\ &\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + i\left[\vec{\nabla}\times\vec{H}\right] = -\frac{4\pi}{c}\vec{j}_e, \\ &\left(\vec{\nabla}\cdot\vec{H}\right) = 0, \\ &\frac{1}{c}\frac{\partial\vec{H}}{\partial t} - i\left[\vec{\nabla}\times\vec{E}\right] = 0. \end{aligned} \tag{4.57}$$

Chapter 5. Gravitational field

The analogy between electromagnetic and gravitational fields was discussed by many researchers starting from J.C.Maxwell and O.Heaviside [31,32]. This analogy motivated many attempts to reformulate the equations of Newtonian gravitation in the form similar to the Maxwell equations in electrodynamics. Such approach is based on two general assumptions. First is the existence of gravitomagnetic field connected with moving masses. Second is the hypothesis that the speed of gravitational field propagation is equal to the speed of light. These assumptions enable the formulation of phenomenological Maxwell-like equations for gravitational field [33, 34]. On the other hand, recently it was shown that linearized weak field equations of general relativity can be represented as the set of Maxwell-like equations for the vectorial gravitational field.

The application of hypercomplex numbers in the theory of weak gravitational field is considered in [35,36].

5.1. Linear equations of gravitational field in flat space-time

As is well known, Einstein's equation for gravitational field is written as [26]:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} , \tag{5.1}$$

where $R_{\alpha\beta}$ is Ricci curvature tensor, $g_{\alpha\beta}$ is the metric tensor, G is the gravitational constant, $T_{\alpha\beta}$ is the tensor of energy-momentum of matter (Greek indexes are 0,1,2,3, Latin indexes are 1,2,3). In linear approximation this equation has the following form [37-39]:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta} , \tag{5.2}$$

where $\bar{h}_{\alpha\beta}$ is the deviation from Minkovski metric tensor $\eta_{\alpha\beta}$, defined by following relations:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},$$
$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h, \tag{5.3}$$
$$h \equiv \eta^{\alpha\beta} h_{\alpha\beta}.$$

The deviation $\bar{h}_{\alpha\beta}$ ($\left|\bar{h}_{\alpha\beta}\right| << 1$) satisfies the gauge condition $\partial \bar{h}_{\alpha\beta} / \partial x_\beta = 0$. Introducing matter density ρ_G and density of matter current $\vec{j}_G$, according the relations

$$T_{00} = \rho_G c^2, \tag{5.4}$$
$$T_{0n} = j_{Gn} c, \tag{5.5}$$

as well as scalar φ_G and vector $\vec{A}_G$ potentials according

$$\bar{h}_{00} = \frac{4}{c^2}\varphi_G, \tag{5.6}$$
$$\bar{h}_{0n} = \frac{4}{c^2} A_{Gn}, \tag{5.7}$$

the equation (5.2) can be represented as the set of wave equations for gravitational potentials:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi_G = -4\pi G \rho_G, \tag{5.8}$$
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{A}_G = -\frac{4\pi}{c} G\, \vec{j}_G, \tag{5.9}$$

with gauge condition

$$\frac{1}{c}\frac{\partial \varphi_G}{\partial t} + \left(\vec{\nabla} \cdot \vec{A}_G\right) = 0. \tag{5.10}$$

The analogy with electrodynamics is evident. It allows one to introduce the gravitoelectric $\vec{E}_G$ and gravitomagnetic $\vec{H}_G$ fields:

$$\vec{E}_G = -\frac{1}{c}\frac{\partial \vec{A}_G}{\partial t} - \vec{\nabla}\varphi_G, \tag{5.11}$$
$$\vec{H}_G = \left[\vec{\nabla} \times \vec{A}_G\right]. \tag{5.12}$$

These variables satisfy the equations similar to Maxwell's equations. In the Heaviside-Gibbs algebra these equations are written as

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{E}_G\right) = -4\pi G\rho_G,\\
&\left(\vec{\nabla}\cdot\vec{H}_G\right) = 0,\\
&\left[\vec{\nabla}\times\vec{E}_G\right] = -\frac{1}{c}\frac{\partial\vec{H}_G}{\partial t},\\
&\left[\vec{\nabla}\times\vec{H}_G\right] = \frac{1}{c}\frac{\partial\vec{E}_G}{\partial t} - \frac{4\pi}{c}\vec{j}_G.
\end{aligned} \tag{5.13}$$

The same equations can be written in sedeonic algebra. In this case the field strengths are defined as

$$\vec{E}_G = -\frac{1}{c}\frac{\partial\vec{A}_G}{\partial t} - \vec{\nabla}\varphi_G, \tag{5.14}$$

$$\vec{H}_G = -i\left[\vec{\nabla}\times\vec{A}_G\right]. \tag{5.15}$$

Then the equations for the gravitational field are written in the following sedeon form:

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{E}_G\right) = -4\pi G\rho_G,\\
&\left(\vec{\nabla}\cdot\vec{H}_G\right) = 0,\\
&-i\left[\vec{\nabla}\times\vec{E}_G\right] = -\frac{1}{c}\frac{\partial\vec{H}_G}{\partial t},\\
&-i\left[\vec{\nabla}\times\vec{H}_G\right] = \frac{1}{c}\frac{\partial\vec{E}_G}{\partial t} - \frac{4\pi}{c}\vec{j}_G.
\end{aligned} \tag{5.16}$$

The gravitational field equations (5.16) differ from the equations for the electromagnetic field (4.17) by sign in front of the terms describing the sources of the field.

Chapter 6. Gravitoelectromagnetism

In this section we develop the sedeonic approach to the formulation of equations for generalized massless gravito-electromagnetic (GE) field describing simultaneously electromagnetism and weak gravity [18].

6.1. Generalized Newton-Coulomb law

It is known that Coulomb's law for the force of electrical interaction between two charged point bodies is written as follows:

$$\vec{F}_{e12} = \frac{q_{e1} q_{e2}}{r_{12}^3} \vec{r}_{12} , \tag{6.1}$$

where q_{e1} and q_{e2} are electrical charges; $\vec{r}_{12}$ is a vector directed from body 1 to body 2; r_{12} is the separation between point bodies, which is equal to modulus of $\vec{r}_{12}$. For a symmetric description of electromagnetic and gravitational phenomena, we introduce the gravitational charge q_g, considered previously in [33, 40]:

$$q_g = \sqrt{G} m , \tag{6.2}$$

where G is gravitational constant, m is a mass of gravitating body. Then Newton's law for gravitational force between two point bodies can be written in the form of Coulomb's law:

$$\vec{F}_{g12} = -\frac{q_{g1} q_{g2}}{r_{12}^3} \vec{r}_{12} . \tag{6.3}$$

Simultaneous consideration of gravitational and electric fields leads us to another symmetry connected with charge conjugation. From the algebraic point of view this symmetry can be taken into account by introducing additional scalar units associated with electrical and gravitational charges. Let us denote the electric unit by symbol $\boldsymbol{\varepsilon}_e$. This value is changed (in sign) under electrical charge conjugation. Analogously the gravitational unit $\boldsymbol{\varepsilon}_g$ is changed (in sign) under gravitational charge conjugation. Since in the classical gravito-electrodynamics there is no direct interaction between gravitational and electrical charges, the rules of multiplication for units $\boldsymbol{\varepsilon}_e$ and $\boldsymbol{\varepsilon}_g$ should be chosen in accordance with Table 3.

Table 3. The rules of multiplication for $\boldsymbol{\varepsilon}_e$ and $\boldsymbol{\varepsilon}_g$ units.

	$\boldsymbol{\varepsilon}_e$	$\boldsymbol{\varepsilon}_g$
$\boldsymbol{\varepsilon}_e$	1	0
$\boldsymbol{\varepsilon}_g$	0	1

Besides, we suppose the anticommutativity of these units and assume that the priority of commutation is higher than multiplication, so that:

$$\boldsymbol{\varepsilon}_e\boldsymbol{\varepsilon}_g = -\boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_e, \tag{6.4}$$

and

$$\boldsymbol{\varepsilon}_e\boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_e = -\boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_e\boldsymbol{\varepsilon}_e = -\boldsymbol{\varepsilon}_g\,. \tag{6.5}$$

Following this approach, the generalized gravito-electromagnetic charge Q can be presented as

$$Q = \boldsymbol{\varepsilon}_e q_e - i\boldsymbol{\varepsilon}_g q_g\,. \tag{6.6}$$

Then generalized Newton - Coulomb law can be written in the following symmetric form:

$$\vec{F}_{12} = \frac{Q_1 Q_2}{r_{12}^3}\vec{r}_{12}\,. \tag{6.7}$$

Indeed, using (6.6) and (6.7) we obtain correct expression for the force between two massive electrically charged point bodies

$$\vec{F}_{12} = \frac{q_{e1}q_{e2}}{r_{12}^3}\vec{r}_{12} - \frac{q_{g1}q_{g2}}{r_{12}^3}\vec{r}_{12}\,. \tag{6.8}$$

Using algebra of gravitoelectrical units we can introduce the operations of electrical charge conjugations ($\hat{I}_e$), gravitational charge conjugation ($\hat{I}_g$), and electrogravitational charge conjugation ($\hat{I}_{eg}$) as

$$\hat{I}_e Q = \boldsymbol{\varepsilon}_g Q \boldsymbol{\varepsilon}_g = -\boldsymbol{\varepsilon}_e q_e - i\boldsymbol{\varepsilon}_g q_g\,, \tag{6.9}$$

$$\hat{I}_g Q = \boldsymbol{\varepsilon}_e Q \boldsymbol{\varepsilon}_e = \boldsymbol{\varepsilon}_e q_e + i\boldsymbol{\varepsilon}_g q_g\,, \tag{6.10}$$

$$\hat{I}_{eg} Q = \boldsymbol{\varepsilon}_g\boldsymbol{\varepsilon}_e Q \boldsymbol{\varepsilon}_e\boldsymbol{\varepsilon}_g = -\boldsymbol{\varepsilon}_e q_e + i\boldsymbol{\varepsilon}_g q_g\,. \tag{6.11}$$

6.2. Generalized sedeonic equations for GE field

The sedeonic formalism enables the representation of gravitational and electromagnetic fields as one unified gravito-electromagnetic field. Let us consider the potential of GE field in the following sedeonic form:

$$\tilde{\mathbf{W}} = i\mathbf{e}_t\boldsymbol{\varepsilon}_e\varphi_e + \mathbf{e}_r\boldsymbol{\varepsilon}_e\vec{A}_e + i\left(i\mathbf{e}_t\boldsymbol{\varepsilon}_g\varphi_g + \mathbf{e}_r\boldsymbol{\varepsilon}_g\vec{A}_g\right), \tag{6.12}$$

where $\varphi_e, \vec{A}_e, \varphi_g$ and $\vec{A}_g$ are scalar and vector potentials of electromagnetic (index e) and gravitational (index g) fields. Hereafter we mean that electrical values contain $\boldsymbol{\varepsilon}_e$ and gravitational values contain $\boldsymbol{\varepsilon}_g$ units, but we will omit them to simplify farther expressions.

The generalized sedeonic second-order equation for massless field can be written in the following form:

$$\left(i\mathbf{e}_t\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_r\vec{\nabla}\right)\left(i\mathbf{e}_t\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_r\vec{\nabla}\right)\tilde{\mathbf{W}} = \tilde{\mathbf{J}}\,. \tag{6.13}$$

Let us also consider the sedeonic source of GE field

$$\tilde{\mathbf{J}} = -4\pi\left(i\mathbf{e}_t\rho_e + \mathbf{e}_r\frac{1}{c}\vec{j}_e\right) + 4\pi i\left(i\mathbf{e}_t\rho_g + \mathbf{e}_r\frac{1}{c}\vec{j}_g\right), \tag{6.14}$$

where ρ_e is a volume density of electrical charge; $\vec{j}_e$ is a density of electrical current; ρ_g is a volume density of gravitational charge; $\vec{j}_g$ is a density of gravitational current. In expanded form equation (6.13) is written as

$$\begin{aligned}&\left(i\mathbf{e}_t\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_r\vec{\nabla}\right)\left(i\mathbf{e}_t\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_r\vec{\nabla}\right)\left(i\mathbf{e}_t\varphi_e + \mathbf{e}_r\vec{A}_e + i\left(i\mathbf{e}_t\varphi_g + \mathbf{e}_r\vec{A}_g\right)\right)\\ &= -4\pi\left(i\mathbf{e}_t\rho_e + \mathbf{e}_r\frac{1}{c}\vec{j}_e\right) + 4\pi i\left(i\mathbf{e}_t\rho_g + \mathbf{e}_r\frac{1}{c}\vec{j}_g\right).\end{aligned} \tag{6.15}$$

This equation describes simultaneously electromagnetic and gravitational fields. Performing sedeonic multiplication of operators in the left part of (6.15) we get the system of wave equations for the components of GE potentials

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi_e = 4\pi\rho_e\,, \tag{6.16}$$

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{A}_e = 4\pi\frac{1}{c}\vec{j}_e, \tag{6.17}$$

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\varphi_g = -4\pi\rho_g, \tag{6.18}$$

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{A}_g = -4\pi\frac{1}{c}\vec{j}_g. \tag{6.19}$$

On the other hand, sedeonic equation (6.15) can be represented as the system of first-order Maxwell equations for electromagnetic and gravitational fields. Let us consider the sequential action of operators in the left part of equation (6.15). After first operator we get

$$\begin{aligned}
&\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(i\mathbf{e_t}\varphi_e+\mathbf{e_r}\vec{A}_e+i\left(i\mathbf{e_t}\varphi_g+\mathbf{e_r}\vec{A}_g\right)\right)= \\
&=-\frac{1}{c}\frac{\partial\varphi_e}{\partial t}-\mathbf{e_{tr}}\frac{1}{c}\frac{\partial\vec{A}_e}{\partial t}-\mathbf{e_{tr}}\vec{\nabla}\varphi_e-\left(\vec{\nabla}\cdot\vec{A}_e\right)-\left[\vec{\nabla}\times\vec{A}_e\right] \\
&+i\left\{-\frac{1}{c}\frac{\partial\varphi_g}{\partial t}-\mathbf{e_{tr}}\frac{1}{c}\frac{\partial\vec{A}_g}{\partial t}-\mathbf{e_{tr}}\vec{\nabla}\varphi_g-\left(\vec{\nabla}\cdot\vec{A}_g\right)-\left[\vec{\nabla}\times\vec{A}_g\right]\right\}.
\end{aligned} \tag{6.20}$$

This expression allows us to introduce scalar and vector field strengths according to the following definitions:

$$\begin{aligned}
&f_e=\frac{1}{c}\frac{\partial\varphi_e}{\partial t}+\left(\vec{\nabla}\cdot\vec{A}_e\right), \\
&\vec{E}_e=-\vec{\nabla}\varphi_e-\frac{1}{c}\frac{\partial\vec{A}_e}{\partial t}, \\
&\vec{H}_e=-i\left[\vec{\nabla}\times\vec{A}_e\right], \\
&f_g=\frac{1}{c}\frac{\partial\varphi_g}{\partial t}+\left(\vec{\nabla}\cdot\vec{A}_g\right), \\
&\vec{E}_g=-\vec{\nabla}\varphi_g-\frac{1}{c}\frac{\partial\vec{A}_g}{\partial t}, \\
&\vec{H}_g=-i\left[\vec{\nabla}\times\vec{A}_g\right].
\end{aligned} \tag{6.21}$$

Using the definitions (6.21), the expression (6.20) can be rewritten in the following form:

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} \right)\left(i\mathbf{e_t}\varphi_e + \mathbf{e_r}\vec{A}_e + i\left(i\mathbf{e_t}\varphi_g + \mathbf{e_r}\vec{A}_g \right)\right)$$
$$= -f_e + \mathbf{e_{tr}}\vec{E}_e - i\vec{H}_e + i\left(-f_g + \mathbf{e_{tr}}\vec{E}_g - i\vec{H}_g \right). \qquad (6.22)$$

Then the second-order wave equation (6.15) can be represented as

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} \right)\left(-f_e + \mathbf{e_{tr}}\vec{E}_e - i\vec{H}_e + i\left(-f_g + \mathbf{e_{tr}}\vec{E}_g - i\vec{H}_g \right)\right)$$
$$= -4\pi\left(i\mathbf{e_t}\rho_e + \mathbf{e_r}\frac{1}{c}\vec{j}_e \right) + 4\pi i\left(i\mathbf{e_t}\rho_g + \mathbf{e_r}\frac{1}{c}\vec{j}_g \right). \qquad (6.23)$$

Applying the operator $\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} \right)$ to both parts of expression (6.23) one can obtain the second-order wave equations for the field strengths in the following form:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \right) f_e = \frac{4\pi}{c}\left\{ \frac{\partial \rho_e}{\partial t} + \left(\vec{\nabla}\cdot\vec{j}_e \right) \right\}, \qquad (6.24)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \right) \vec{E}_e = -4\pi\vec{\nabla}\rho_e - \frac{4\pi}{c^2}\frac{\partial \vec{j}_e}{\partial t}, \qquad (6.25)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \right) \vec{H}_e = -i\frac{4\pi}{c}\left[\vec{\nabla}\times\vec{j}_e \right], \qquad (6.26)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \right) f_g = -\frac{4\pi}{c}\left\{ \frac{\partial \rho_g}{\partial t} + \left(\vec{\nabla}\cdot\vec{j}_g \right) \right\}, \qquad (6.27)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \right) \vec{E}_g = 4\pi\vec{\nabla}\rho_g + \frac{4\pi}{c^2}\frac{\partial \vec{j}_g}{\partial t}, \qquad (6.28)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \right) \vec{H}_g = i\frac{4\pi}{c}\left[\vec{\nabla}\times\vec{j}_g \right]. \qquad (6.29)$$

Assuming the conservation of electrical and gravitational charges we have

$$\frac{\partial \rho_e}{\partial t} + \left(\vec{\nabla}\cdot\vec{j}_e \right) = 0, \qquad (6.30)$$

$$\frac{\partial \rho_g}{\partial t} + \left(\vec{\nabla}\cdot\vec{j}_g \right) = 0, \qquad (6.31)$$

and we can take the scalar fields f_e and f_g equal to zero. This is equivalent to the Lorentz gauge conditions (see expressions (6.21)):

$$f_e = \frac{1}{c}\frac{\partial \varphi_e}{\partial t} + (\vec{\nabla} \cdot \vec{A}_e) = 0\,, \tag{6.32}$$

$$f_g = \frac{1}{c}\frac{\partial \varphi_g}{\partial t} + (\vec{\nabla} \cdot \vec{A}_g) = 0\,. \tag{6.33}$$

Taking into account these gauge conditions the equation (6.22) is rewritten as

$$\begin{aligned}&\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla}\right)\left(i\mathbf{e_t}\varphi_e + \mathbf{e_r}\vec{A}_e + i\left(i\mathbf{e_t}\varphi_g + \mathbf{e_r}\vec{A}_g\right)\right)\\ &= \mathbf{e_{tr}}\vec{E}_e - i\vec{H}_e + i\left(\mathbf{e_{tr}}\vec{E}_g - i\vec{H}_g\right),\end{aligned} \tag{6.34}$$

and generalized equation (6.15) takes the following form:

$$\begin{aligned}&\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla}\right)\left(\mathbf{e_{tr}}\vec{E}_e - i\vec{H}_e + i\left(\mathbf{e_{tr}}\vec{E}_g - i\vec{H}_g\right)\right)\\ &= -4\pi\left(i\mathbf{e_t}\rho_e + \mathbf{e_r}\frac{1}{c}\vec{j}_e\right) + 4\pi i\left(i\mathbf{e_t}\rho_g + \mathbf{e_r}\frac{1}{c}\vec{j}_g\right).\end{aligned} \tag{6.35}$$

Performing sedeonic multiplication in the left part of equation (6.35) and separating terms with different space-time properties, we obtain the system of Maxwell equations for GE field

$$\begin{aligned}&-i\left[\vec{\nabla}\times\left(\vec{E}_e + i\vec{E}_g\right)\right] = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{H}_e + i\vec{H}_g\right),\\ &-i\left[\vec{\nabla}\times\left(\vec{H}_e + i\vec{H}_g\right)\right] = \frac{4\pi}{c}\left(\vec{j}_e - i\,\vec{j}_g\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E}_e + i\vec{E}_g\right),\\ &\left(\vec{\nabla}\cdot\left(\vec{E}_e + i\vec{E}_g\right)\right) = 4\pi\left(\rho_e - i\rho_g\right),\\ &\left(\vec{\nabla}\cdot\left(\vec{H}_e + i\vec{H}_g\right)\right) = 0.\end{aligned} \tag{6.36}$$

Separating terms with different charge ($\boldsymbol{\varepsilon}_e$ and $\boldsymbol{\varepsilon}_g$) properties, we obtain two systems of Maxwell equations for electromagnetic and gravitational fields. For electromagnetic field we have

$$
\begin{aligned}
&-i\left[\vec{\nabla}\times\vec{E}_e\right] = -\frac{1}{c}\frac{\partial \vec{H}_e}{\partial t},\\
&-i\left[\vec{\nabla}\times\vec{H}_e\right] = \frac{4\pi}{c}\vec{j}_e + \frac{1}{c}\frac{\partial \vec{E}_e}{\partial t},\\
&\left(\vec{\nabla}\cdot\vec{E}_e\right) = 4\pi\rho_e,\\
&\left(\vec{\nabla}\cdot\vec{H}_e\right) = 0.
\end{aligned}
\tag{6.37}
$$

On the other hand for gravitational field we obtain

$$
\begin{aligned}
&-i\left[\vec{\nabla}\times\vec{E}_g\right] = -\frac{1}{c}\frac{\partial \vec{H}_g}{\partial t},\\
&-i\left[\vec{\nabla}\times\vec{H}_g\right] = -\frac{4\pi}{c}\vec{j}_g + \frac{1}{c}\frac{\partial \vec{E}_g}{\partial t},\\
&\left(\vec{\nabla}\cdot\vec{E}_g\right) = -4\pi\rho_g,\\
&\left(\vec{\nabla}\cdot\vec{H}_g\right) = 0.
\end{aligned}
\tag{6.38}
$$

Thus, we have shown that the generalized sedeonic equation (6.15) correctly describes the unified GE field. Further, we will assume the performing of gauge conditions (6.32) and (6.33).

6.3. Relations for energy and momentum of GE field

The sedeonic wave equation allows one to derive the generalized Pointing theorem for unified GE field. Multiplying the expression (6.35) on the sedeon $\mathbf{e}_{\mathbf{tr}}\vec{E}_e - i\vec{H}_e + i\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_g - i\vec{H}_g\right)$ from the left, we have the following equation:

$$
\begin{aligned}
&\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_e - i\vec{H}_e + i\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_g - i\vec{H}_g\right)\right)\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_e - i\vec{H}_e + i\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_g - i\vec{H}_g\right)\right)\\
&= -4\pi\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_e - i\vec{H}_e + i\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_g - i\vec{H}_g\right)\right)\left(i\mathbf{e}_{\mathbf{t}}\rho_e + \mathbf{e}_{\mathbf{r}}\frac{1}{c}\vec{j}_e - i\left(i\mathbf{e}_{\mathbf{t}}\rho_g + \mathbf{e}_{\mathbf{r}}\frac{1}{c}\vec{j}_g\right)\right).
\end{aligned}
\tag{6.39}
$$

Performing the sedeonic multiplication in (6.39), we obtain

$$
\begin{aligned}
&-i\mathbf{e_t}\left\{\frac{1}{2c}\frac{\partial}{\partial t}\left\{\left(\vec{E}_e+i\vec{E}_g\right)^2+\left(\vec{H}_e+i\vec{H}_g\right)^2\right\}-i\left(\vec{\nabla}\cdot\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right)\right\}\\
&+\mathbf{e_r}\left\{i\frac{1}{c}\left(\left(\vec{E}_e+i\vec{E}_g\right)\cdot\frac{\partial\left(\vec{H}_e+i\vec{H}_g\right)}{\partial t}\right)-i\frac{1}{c}\left(\left(\vec{H}_e+i\vec{H}_g\right)\cdot\frac{\partial\left(\vec{E}_e+i\vec{E}_g\right)}{\partial t}\right)\right.\\
&\left.+\left(\left(\vec{E}_e+i\vec{E}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{E}_e+i\vec{E}_g\right)\right]\right)+\left(\left(\vec{H}_e+i\vec{H}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right)\right\}\\
&+\mathbf{e_t}\left\{-i\frac{1}{c}\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\frac{\partial\left(\vec{E}_e+i\vec{E}_g\right)}{\partial t}\right]-i\frac{1}{c}\left[\left(\vec{H}_e+i\vec{H}_g\right)\times\frac{\partial\left(\vec{H}_e+i\vec{H}_g\right)}{\partial t}\right]\right.\\
&+\left(\vec{E}_e+i\vec{E}_g\right)\left(\vec{\nabla}\cdot\left(\vec{H}_e+i\vec{H}_g\right)\right)-\left(\vec{H}_e+i\vec{H}_g\right)\left(\vec{\nabla}\cdot\left(\vec{E}_e+i\vec{E}_g\right)\right)\\
&\left.+\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\left[\vec{\nabla}\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right]-\left[\left(\vec{H}_e+i\vec{H}_g\right)\times\left[\vec{\nabla}\times\left(\vec{E}_e+i\vec{E}_g\right)\right]\right]\right\}\\
&+\mathbf{e_r}\left\{i\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\left(\vec{H}_e+i\vec{H}_g\right)\right]-\frac{1}{2}\vec{\nabla}\left\{\left(\vec{E}_e+i\vec{E}_g\right)^2+\left(\vec{H}_e+i\vec{H}_g\right)^2\right\}\right.\\
&\left.+\left(\vec{\nabla}\cdot\left(\vec{E}_e+i\vec{E}_g\right)\right)\left(\vec{E}_e+i\vec{E}_g\right)+\left(\vec{\nabla}\cdot\left(\vec{H}_e+i\vec{H}_g\right)\right)\left(\vec{H}_e+i\vec{H}_g\right)\right\}\\
&=i\mathbf{e_t}\frac{4\pi}{c}\left(\left(\vec{E}_e+i\vec{E}_g\right)\cdot\left(\vec{j}_e-i\vec{j}_g\right)\right)+i\mathbf{e_r}\frac{4\pi}{c}\left(\left(\vec{H}_e+i\vec{H}_g\right)\cdot\left(\vec{j}_e-i\vec{j}_g\right)\right)\\
&-4\pi\mathbf{e_t}\left\{\left(\rho_e-i\rho_g\right)\left(\vec{H}_e+i\vec{H}_g\right)-i\frac{1}{c}\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\left(\vec{j}_e-i\vec{j}_g\right)\right]\right\}\\
&+4\pi\mathbf{e_r}\left\{\left(\rho_e-i\rho_g\right)\left(\vec{E}_e+i\vec{E}_g\right)+i\frac{1}{c}\left[\left(\vec{H}_e+i\vec{H}_g\right)\times\left(\vec{j}_e-i\vec{j}_g\right)\right]\right\}.
\end{aligned}
\tag{6.40}
$$

Note that in this expression and further the operator $\vec{\nabla}$ acts on all right expression. For example, for any vectors $\vec{A}$ and $\vec{B}$ we have

$$\left(\vec{\nabla}\cdot\vec{A}\right)\vec{B}=\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)+\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}\,. \tag{6.41}$$

Equating in (6.40) the components with different space-time properties we get the following equations for the GE field strengths:

$$
\begin{aligned}
&\frac{1}{8\pi}\frac{\partial}{\partial t}\left\{\left(\vec{E}_e+i\vec{E}_g\right)^2+\left(\vec{H}_e+i\vec{H}_g\right)^2\right\}-i\frac{c}{4\pi}\left(\vec{\nabla}\cdot\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right)\\
&+\left(\left(\vec{E}_e+i\vec{E}_g\right)\cdot\left(\vec{j}_e-i\,\vec{j}_g\right)\right)=0,
\end{aligned}
\tag{6.42}
$$

$$\frac{1}{4\pi}\left\{\left(\left(\vec{E}_e+i\vec{E}_g\right)\cdot\frac{\partial\left(\vec{H}_e+i\vec{H}_g\right)}{\partial t}\right)-\left(\left(\vec{H}_e+i\vec{H}_g\right)\cdot\frac{\partial\left(\vec{E}_e+i\vec{E}_g\right)}{\partial t}\right)\right\}$$

$$-i\frac{c}{4\pi}\left\{\left(\left(\vec{E}_e+i\vec{E}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{E}_e+i\vec{E}_g\right)\right]\right)+\left(\left(\vec{H}_e+i\vec{H}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right)\right\} \quad (6.43)$$

$$-\left(\left(\vec{H}_e+i\vec{H}_g\right)\cdot\left(\vec{j}_e-i\,\vec{j}_g\right)\right)=0,$$

$$-i\frac{1}{4\pi}\left\{\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\frac{\partial\left(\vec{E}_e+i\vec{E}_g\right)}{\partial t}\right]+\left[\left(\vec{H}_e+i\vec{H}_g\right)\times\frac{\partial\left(\vec{H}_e+i\vec{H}_g\right)}{\partial t}\right]\right\}$$

$$+\frac{c}{4\pi}\left\{\left(\vec{E}_e+i\vec{E}_g\right)\left(\vec{\nabla}\cdot\left(\vec{H}_e+i\vec{H}_g\right)\right)-\left(\vec{H}_e+i\vec{H}_g\right)\left(\vec{\nabla}\cdot\left(\vec{E}_e+i\vec{E}_g\right)\right)\right\} \quad (6.44)$$

$$+\frac{c}{4\pi}\left\{\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\left[\vec{\nabla}\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right]-\left[\left(\vec{H}_e+i\vec{H}_g\right)\times\left[\vec{\nabla}\times\left(\vec{E}_e+i\vec{E}_g\right)\right]\right]\right\}$$

$$+c\left(\vec{H}_e+i\vec{H}_g\right)\left(\rho_e-i\rho_g\right)-i\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\left(\vec{j}_e-i\vec{j}_g\right)\right]=0,$$

$$-i\frac{1}{4\pi}\frac{\partial}{\partial t}\left[\left(\vec{E}_e+i\vec{E}_g\right)\times\left(\vec{H}_e+i\vec{H}_g\right)\right]+\frac{c}{8\pi}\vec{\nabla}\left\{\left(\vec{E}_e+i\vec{E}_g\right)^2+\left(\vec{H}_e+i\vec{H}_g\right)^2\right\}$$

$$-\frac{c}{4\pi}\left\{\left(\vec{\nabla}\cdot\left(\vec{E}_e+i\vec{E}_g\right)\right)\left(\vec{E}_e+i\vec{E}_g\right)+\left(\vec{\nabla}\cdot\left(\vec{H}_e+i\vec{H}_g\right)\right)\left(\vec{H}_e+i\vec{H}_g\right)\right\} \quad (6.45)$$

$$+c\left(\rho_e-i\rho_g\right)\left(\vec{E}_e+i\vec{E}_g\right)-i\left[\left(\vec{H}_e+i\vec{H}_g\right)\times\left(\vec{j}_e-i\vec{j}_g\right)\right]=0.$$

Finally, separating the values with different space-time properties and taking into account that $\boldsymbol{\varepsilon}_e\boldsymbol{\varepsilon}_g=0$, we get

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left\{\vec{E}_e^2+\vec{H}_e^2-\vec{E}_g^2-\vec{H}_g^2\right\}$$

$$-i\frac{c}{4\pi}\left\{\left(\vec{\nabla}\cdot\left[\vec{E}_e\times\vec{H}_e\right]\right)-\left(\vec{\nabla}\cdot\left[\vec{E}_g\times\vec{H}_g\right]\right)\right\} \quad (6.46)$$

$$+\left\{\left(\vec{E}_e\cdot\vec{j}_e\right)+\left(\vec{E}_g\cdot\vec{j}_g\right)\right\}=0.$$

$$
\begin{aligned}
&\frac{1}{8\pi}\vec{\nabla}\left\{\vec{E}_e^2+\vec{H}_e^2-\vec{E}_g^2-\vec{H}_g^2\right\}\\
&-i\frac{1}{4\pi c}\frac{\partial}{\partial t}\left\{\left[\vec{E}_e\times\vec{H}_e\right]-\left[\vec{E}_g\times\vec{H}_g\right]\right\}\\
&-\frac{1}{4\pi}\left\{\left(\vec{\nabla}\cdot\vec{E}_e\right)\vec{E}_e+\left(\vec{\nabla}\cdot\vec{H}_e\right)\vec{H}_e\right\}\\
&+\frac{1}{4\pi}\left\{\left(\vec{\nabla}\cdot\vec{E}_g\right)\vec{E}_g+\left(\vec{\nabla}\cdot\vec{H}_g\right)\vec{H}_g\right\}\\
&+\left\{\rho_e\vec{E}_e+\rho_g\vec{E}_g\right\}+i\left\{\left[\vec{H}_e\times\vec{j}_e\right]+\left[\vec{H}_g\times\vec{j}_g\right]\right\}=0,
\end{aligned}
\tag{6.47}
$$

$$
\begin{aligned}
&\frac{1}{4\pi}\left\{\left(\vec{E}_e\cdot\frac{\partial\vec{H}_e}{\partial t}\right)-\left(\vec{H}_e\cdot\frac{\partial\vec{E}_e}{\partial t}\right)\right\}\\
&+\frac{1}{4\pi}\left\{\left(\vec{H}_g\cdot\frac{\partial\vec{E}_g}{\partial t}\right)-\left(\vec{E}_g\cdot\frac{\partial\vec{H}_g}{\partial t}\right)\right\}\\
&-i\frac{c}{4\pi}\left\{\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)+\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)\right\}\\
&+i\frac{c}{4\pi}\left\{\left(\vec{E}_g\cdot\left[\vec{\nabla}\times\vec{E}_g\right]\right)+\left(\vec{H}_g\cdot\left[\vec{\nabla}\times\vec{H}_g\right]\right)\right\}\\
&-\left\{\left(\vec{H}_e\cdot\vec{j}_e\right)+\left(\vec{H}_g\cdot\vec{j}_g\right)\right\}=0,
\end{aligned}
\tag{6.48}
$$

$$
\begin{aligned}
&-i\frac{1}{4\pi}\left\{\left[\vec{E}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]-\left[\vec{E}_g\times\frac{\partial\vec{E}_g}{\partial t}\right]+\left[\vec{H}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]-\left[\vec{H}_g\times\frac{\partial\vec{H}_g}{\partial t}\right]\right\}\\
&+\frac{c}{4\pi}\left\{\vec{E}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)-\vec{E}_g\left(\vec{\nabla}\cdot\vec{H}_g\right)-\vec{H}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)+\vec{H}_g\left(\vec{\nabla}\cdot\vec{E}_g\right)\right\}\\
&+\frac{c}{4\pi}\left\{\left[\vec{E}_e\times\left[\vec{\nabla}\times\vec{H}_e\right]\right]-\left[\vec{H}_e\times\left[\vec{\nabla}\times\vec{E}_e\right]\right]\right\}\\
&+\frac{c}{4\pi}\left\{\left[\vec{H}_g\times\left[\vec{\nabla}\times\vec{E}_g\right]\right]-\left[\vec{E}_g\times\left[\vec{\nabla}\times\vec{H}_g\right]\right]\right\}\\
&+c\left\{\vec{H}_e\rho_e+\vec{H}_g\rho_g\right\}-i\left\{\left[\vec{E}_e\times\vec{j}_e\right]+\left[\vec{E}_g\times\vec{j}_g\right]\right\}=0.
\end{aligned}
\tag{6.49}
$$

The expression (6.46) is the generalized Pointing theorem for the GE field. The value w

$$w=\frac{1}{8\pi}\left\{\vec{E}_e^2+\vec{H}_e^2-\vec{E}_g^2-\vec{H}_g^2\right\} \tag{6.50}$$

plays the role of volume density of GE field energy, while vector $\vec{P}$

$$\vec{P}=-i\frac{c}{4\pi}\left\{\left[\vec{E}_e\times\vec{H}_e\right]-\left[\vec{E}_g\times\vec{H}_g\right]\right\} \tag{6.51}$$

plays the role of Pointing vector of GE field. Besides, the vector

$$\vec{f}_L=\rho_e\vec{E}_e+i\left[\vec{H}_e\times\vec{j}_e\right]+\rho_g\vec{E}_g+i\left[\vec{H}_g\times\vec{j}_g\right]$$

is the volume density of generalized Lorentz force.

6.4. Lorentz invariants of GE field

The sedeonic algebra allows one to obtain relations for the Lorentz invariants of GE field. Let us multiply expression (6.35) from the left on sedeon

$$\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_e+i\vec{H}_e-i\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_g+i\vec{H}_g\right)\right).$$

As a result, we obtain the following relation:

$$\begin{aligned}&\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_e+i\vec{H}_e-i\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_g+i\vec{H}_g\right)\right)\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_r\vec{\nabla}\right)\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_e-i\vec{H}_e+i\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_g-i\vec{H}_g\right)\right)\\&=-4\pi\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_e+i\vec{H}_e-i\left(\mathbf{e}_{\mathbf{tr}}\vec{E}_g+i\vec{H}_g\right)\right)\left(i\mathbf{e}_{\mathbf{t}}\rho_e+\mathbf{e}_{\mathbf{r}}\frac{1}{c}\vec{j}_e-i\left(i\mathbf{e}_{\mathbf{t}}\rho_g+\mathbf{e}_{\mathbf{r}}\frac{1}{c}\vec{j}_g\right)\right).\end{aligned} \tag{6.52}$$

Then performing sedeonic multiplication, we obtain the following expression:

$$
\begin{aligned}
&-i\mathbf{e_t}\left\{\frac{1}{c}\left(\left(\vec{E}_e-i\vec{E}_g\right)\cdot\frac{\partial\left(\vec{E}_e+i\vec{E}_g\right)}{\partial t}\right)-\frac{1}{c}\left(\left(\vec{H}_e-i\vec{H}_g\right)\cdot\frac{\partial\left(\vec{H}_e+i\vec{H}_g\right)}{\partial t}\right)\right.\\
&\left.+i\left\{\left(\left(\vec{E}_e-i\vec{E}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right)+\left(\left(\vec{H}_e-i\vec{H}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{E}_e+i\vec{E}_g\right)\right]\right)\right\}\right\}\\
&+i\mathbf{e_r}\frac{1}{c}\left\{\left(\left(\vec{E}_e-i\vec{E}_g\right)\cdot\frac{\partial\left(\vec{H}_e+i\vec{H}_g\right)}{\partial t}\right)+\left(\left(\vec{H}_e-i\vec{H}_g\right)\cdot\frac{\partial\left(\vec{E}_e+i\vec{E}_g\right)}{\partial t}\right)\right.\\
&\left.-i\left(\left(\vec{E}_e-i\vec{E}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{E}_e+i\vec{E}_g\right)\right]\right)+i\left(\left(\vec{H}_e-i\vec{H}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right)\right\}\\
&+\mathbf{e_t}\left\{-i\frac{1}{c}\left[\left(\vec{E}_e-i\vec{E}_g\right)\times\frac{\partial\left(\vec{E}_e+i\vec{E}_g\right)}{\partial t}\right]+i\frac{1}{c}\left[\left(\vec{H}_e-i\vec{H}_g\right)\times\frac{\partial\left(\vec{H}_e+i\vec{H}_g\right)}{\partial t}\right]\right.\\
&+\left(\vec{E}_e-i\vec{E}_g\right)\left(\vec{\nabla}\cdot\left(\vec{H}_e+i\vec{H}_g\right)\right)+\left(\vec{H}_e-i\vec{H}_g\right)\left(\vec{\nabla}\cdot\left(\vec{E}_e+i\vec{E}_g\right)\right)\\
&\left.+\left[\left(\vec{E}_e-i\vec{E}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right]+\left[\left(\vec{H}_e-i\vec{H}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{E}_e+i\vec{E}_g\right)\right]\right]\right\} \qquad (6.53)\\
&+\mathbf{e_r}\left\{i\frac{1}{c}\left[\left(\vec{E}_e-i\vec{E}_g\right)\times\frac{\partial\left(\vec{H}_e+i\vec{H}_g\right)}{\partial t}\right]+i\frac{1}{c}\left[\left(\vec{H}_e-i\vec{H}_g\right)\times\frac{\partial\left(\vec{E}_e+i\vec{E}_g\right)}{\partial t}\right]\right.\\
&+\left(\vec{E}_e-i\vec{E}_g\right)\left(\vec{\nabla}\cdot\left(\vec{E}_e+i\vec{E}_g\right)\right)-\left(\vec{H}_e-i\vec{H}_g\right)\left(\vec{\nabla}\cdot\left(\vec{H}_e+i\vec{H}_g\right)\right)\\
&\left.+\left[\left(\vec{E}_e-i\vec{E}_g\right)\times\left[\vec{\nabla}\times\left(\vec{E}_e+i\vec{E}_g\right)\right]\right]-\left[\left(\vec{H}_e-i\vec{H}_g\right)\times\left[\vec{\nabla}\times\left(\vec{H}_e+i\vec{H}_g\right)\right]\right]\right\}\\
&=i\mathbf{e_t}\frac{4\pi}{c}\left(\left(\vec{E}_e-i\vec{E}_g\right)\cdot\left(\vec{j}_e-i\vec{j}_g\right)\right)-i\mathbf{e_r}\frac{4\pi}{c}\left(\left(\vec{H}_e-i\vec{H}_g\right)\cdot\left(\vec{j}_e-i\vec{j}_g\right)\right)\\
&+4\pi\mathbf{e_t}\left\{\left(\rho_e-i\rho_g\right)\left(\vec{H}_e-i\vec{H}_g\right)+i\frac{1}{c}\left[\left(\vec{E}_e-i\vec{E}_g\right)\times\left(\vec{j}_e-i\vec{j}_g\right)\right]\right\}\\
&+4\pi\mathbf{e_r}\left\{\left(\rho_e-i\rho_g\right)\left(\vec{E}_e-i\vec{E}_g\right)-i\frac{1}{c}\left[\left(\vec{H}_e-i\vec{H}_g\right)\times\left(\vec{j}_e-i\vec{j}_g\right)\right]\right\}.
\end{aligned}
$$

Equating in (6.53) the components with different space-time properties, we get the following equations for GE field strengths:

$$
\begin{aligned}
&\frac{1}{4\pi}\left\{\left(\left(\vec{E}_e - i\vec{E}_g\right)\cdot\frac{\partial\left(\vec{E}_e + i\vec{E}_g\right)}{\partial t}\right) - \left(\left(\vec{H}_e - i\vec{H}_g\right)\cdot\frac{\partial\left(\vec{H}_e + i\vec{H}_g\right)}{\partial t}\right)\right\} \\
&+i\frac{c}{4\pi}\left\{\left(\left(\vec{E}_e - i\vec{E}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{H}_e + i\vec{H}_g\right)\right]\right) + \left(\left(\vec{H}_e - i\vec{H}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{E}_e + i\vec{E}_g\right)\right]\right)\right\} \\
&+\left(\left(\vec{E}_e - i\vec{E}_g\right)\cdot\left(\vec{j}_e - i\vec{j}_g\right)\right) = 0,
\end{aligned}
\qquad (6.54)
$$

$$
\begin{aligned}
&\frac{1}{4\pi}\left\{\left(\left(\vec{E}_e - i\vec{E}_g\right)\cdot\frac{\partial\left(\vec{H}_e + i\vec{H}_g\right)}{\partial t}\right) + \left(\left(\vec{H}_e - i\vec{H}_g\right)\cdot\frac{\partial\left(\vec{E}_e + i\vec{E}_g\right)}{\partial t}\right) - \right. \\
&\left. -i\left(\left(\vec{E}_e - i\vec{E}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{E}_e + i\vec{E}_g\right)\right]\right) + i\left(\left(\vec{H}_e - i\vec{H}_g\right)\cdot\left[\vec{\nabla}\times\left(\vec{H}_e + i\vec{H}_g\right)\right]\right)\right\} \\
&+\left(\left(\vec{H}_e - i\vec{H}_g\right)\cdot\left(\vec{j}_e - i\vec{j}_g\right)\right) = 0,
\end{aligned}
\qquad (6.55)
$$

$$
\begin{aligned}
&\frac{c}{4\pi}\left\{\left[\left(\vec{E}_e - i\vec{E}_g\right)\times\left[\vec{\nabla}\times\left(\vec{H}_e + i\vec{H}_g\right)\right]\right] + \left[\left(\vec{H}_e - i\vec{H}_g\right)\times\left[\vec{\nabla}\times\left(\vec{E}_e + i\vec{E}_g\right)\right]\right]\right\} \\
&+\frac{c}{4\pi}\left\{\left(\vec{E}_e - i\vec{E}_g\right)\left(\vec{\nabla}\cdot\left(\vec{H}_e + i\vec{H}_g\right)\right) + \left(\vec{H}_e - i\vec{H}_g\right)\left(\vec{\nabla}\cdot\left(\vec{E}_e + i\vec{E}_g\right)\right)\right\} \\
&-i\frac{1}{4\pi}\left\{\left[\left(\vec{E}_e - i\vec{E}_g\right)\times\frac{\partial\left(\vec{E}_e + i\vec{E}_g\right)}{\partial t}\right] - \left[\left(\vec{H}_e - i\vec{H}_g\right)\times\frac{\partial\left(\vec{H}_e + i\vec{H}_g\right)}{\partial t}\right]\right\} \\
&-c\left(\vec{H}_e - i\vec{H}_g\right)\left(\rho_e - i\rho_g\right) - i\left[\left(\vec{E}_e - i\vec{E}_g\right)\times\left(\vec{j}_e - i\vec{j}_g\right)\right] = 0,
\end{aligned}
\qquad (6.56)
$$

$$
\begin{aligned}
&i\frac{1}{4\pi}\left\{\left[\left(\vec{E}_e - i\vec{E}_g\right)\times\frac{\partial\left(\vec{H}_e + i\vec{H}_g\right)}{\partial t}\right] + \left[\left(\vec{H}_e - i\vec{H}_g\right)\times\frac{\partial\left(\vec{E}_e + i\vec{E}_g\right)}{\partial t}\right]\right\} \\
&+\frac{c}{4\pi}\left\{\left(\vec{E}_e - i\vec{E}_g\right)\left(\vec{\nabla}\cdot\left(\vec{E}_e + i\vec{E}_g\right)\right) - \left(\vec{H}_e - i\vec{H}_g\right)\left(\vec{\nabla}\cdot\left(\vec{H}_e + i\vec{H}_g\right)\right)\right\} \\
&+\frac{c}{4\pi}\left\{\left[\left(\vec{E}_e - i\vec{E}_g\right)\times\left[\vec{\nabla}\times\left(\vec{E}_e + i\vec{E}_g\right)\right]\right] - \left[\left(\vec{H}_e - i\vec{H}_g\right)\times\left[\vec{\nabla}\times\left(\vec{H}_e + i\vec{H}_g\right)\right]\right]\right\} \\
&-c\left(\rho_e - i\rho_g\right)\left(\vec{E}_e - i\vec{E}_g\right) - i\left[\left(\vec{H}_e - i\vec{H}_g\right)\times\left(\vec{j}_e - i\vec{j}_g\right)\right] = 0.
\end{aligned}
\qquad (6.57)
$$

Finally, taking into account that $\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2 = 0$ we get

$$
\begin{aligned}
&\frac{1}{8\pi}\frac{\partial}{\partial t}\left\{\vec{E}_e^2-\vec{H}_e^2+\vec{E}_g^2-\vec{H}_g^2\right\}\\
&+i\frac{c}{4\pi}\left\{\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)+\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)\right\}\\
&+i\frac{c}{4\pi}\left\{\left(\vec{E}_g\cdot\left[\vec{\nabla}\times\vec{H}_g\right]\right)+\left(\vec{H}_g\cdot\left[\vec{\nabla}\times\vec{E}_g\right]\right)\right\}\\
&+\left(\vec{E}_e\cdot\vec{j}_e\right)-\left(\vec{E}_g\cdot\vec{j}_g\right)=0,
\end{aligned}
\tag{6.58}
$$

$$
\begin{aligned}
&\frac{c}{8\pi}\vec{\nabla}\left\{\vec{E}_e^2-\vec{H}_e^2+\vec{E}_g^2-\vec{H}_g^2\right\}\\
&-i\frac{1}{4\pi}\left\{\left[\vec{E}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]+\left[\vec{E}_g\times\frac{\partial\vec{H}_g}{\partial t}\right]+\left[\vec{H}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]+\left[\vec{H}_g\times\frac{\partial\vec{E}_g}{\partial t}\right]\right\}\\
&-\frac{c}{4\pi}\left\{\vec{E}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)+\left(\vec{E}_e\cdot\vec{\nabla}\right)\vec{E}_e+\vec{E}_g\left(\vec{\nabla}\cdot\vec{E}_g\right)+\left(\vec{E}_g\cdot\vec{\nabla}\right)\vec{E}_g\right.\\
&\left.-\vec{H}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)-\left(\vec{H}_e\cdot\vec{\nabla}\right)\vec{H}_e-\vec{H}_g\left(\vec{\nabla}\cdot\vec{H}_g\right)-\left(\vec{H}_g\cdot\vec{\nabla}\right)\vec{H}_g\right\}\\
&+c\left\{\rho_e\vec{E}_e-\rho_g\vec{E}_g\right\}-i\left\{\left[\vec{H}_e\times\vec{j}_e\right]-\left[\vec{H}_g\times\vec{j}_g\right]\right\}=0,
\end{aligned}
\tag{6.59}
$$

$$
\begin{aligned}
&\frac{1}{4\pi}\frac{\partial}{\partial t}\left\{\left(\vec{E}_e\cdot\vec{H}_e\right)+\left(\vec{E}_g\cdot\vec{H}_g\right)\right\}\\
&-i\frac{c}{4\pi}\left\{\left(\vec{E}_e\cdot\left[\vec{\nabla}\times\vec{E}_e\right]\right)-\left(\vec{H}_e\cdot\left[\vec{\nabla}\times\vec{H}_e\right]\right)\right\}\\
&-i\frac{c}{4\pi}\left\{\left(\vec{E}_g\cdot\left[\vec{\nabla}\times\vec{E}_g\right]\right)-\left(\vec{H}_g\cdot\left[\vec{\nabla}\times\vec{H}_g\right]\right)\right\}\\
&+\left(\vec{H}_e\cdot\vec{j}_e\right)-\left(\vec{H}_g\cdot\vec{j}_g\right)=0,
\end{aligned}
\tag{6.60}
$$

$$
\begin{aligned}
&\frac{c}{4\pi}\vec{\nabla}\left\{\left(\vec{E}_e\cdot\vec{H}_e\right)+\left(\vec{E}_g\cdot\vec{H}_g\right)\right\}\\
&-\frac{c}{4\pi}\left\{\vec{E}_e\left(\vec{\nabla}\cdot\vec{H}_e\right)+\vec{E}_g\left(\vec{\nabla}\cdot\vec{H}_g\right)+\vec{H}_e\left(\vec{\nabla}\cdot\vec{E}_e\right)+\vec{H}_g\left(\vec{\nabla}\cdot\vec{E}_g\right)\right.\\
&\left.-\left(\vec{E}_e\cdot\vec{\nabla}\right)\vec{H}_e+\left(\vec{E}_g\cdot\vec{\nabla}\right)\vec{H}_g+\left(\vec{H}_e\cdot\vec{\nabla}\right)\vec{E}_e+\left(\vec{H}_g\cdot\vec{\nabla}\right)\vec{E}_g\right\}\\
&+i\frac{1}{4\pi}\left\{\left[\vec{E}_e\times\frac{\partial\vec{E}_e}{\partial t}\right]+\left[\vec{E}_g\times\frac{\partial\vec{E}_g}{\partial t}\right]-\left[\vec{H}_e\times\frac{\partial\vec{H}_e}{\partial t}\right]-\left[\vec{H}_g\times\frac{\partial\vec{H}_g}{\partial t}\right]\right\}\\
&+c\left\{\vec{H}_e\rho_e-\vec{H}_g\rho_g\right\}-i\left\{\left[\vec{E}_e\times\vec{j}_e\right]-\left[\vec{E}_g\times\vec{j}_g\right]\right\}=0.
\end{aligned}
\tag{6.61}
$$

The expressions (6.58) - (6.61) are the equations for the generalized Lorentz invariants I_1 and I_2 of GE field:

$$I_1 = \vec{E}_e^2 - \vec{H}_e^2 + \vec{E}_g^2 - \vec{H}_g^2 , \tag{6.62}$$

$$I_2 = \left(\vec{E}_e \cdot \vec{H}_e\right) + \left(\vec{E}_g \cdot \vec{H}_g\right) . \tag{6.63}$$

Chapter 7. Relativistic quantum mechanics

7.1. Sedeonic wave equation for particles in an external fields

Let us consider a relativistic quantum particle, which is described by the sedeonic wave equation (see (3.6)):

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\tilde{\boldsymbol{\psi}}=0, \tag{7.1}$$

where wave function is space-time sedeon

$$\tilde{\boldsymbol{\psi}}(\vec{r},t)=\boldsymbol{\psi_0}(\vec{r},t)+\vec{\boldsymbol{\psi}}(\vec{r},t). \tag{7.2}$$

In the equation (7.1) the elements of sedeonic basis $\mathbf{e_n}$ and $\mathbf{a_m}$ play the role of space-time operators, which transform the space-time structure of the wave function $\tilde{\boldsymbol{\psi}}$ according the multiplication rules. For example, let us consider the action of $\mathbf{a_3}$ operator. The wave function can be presented in $\mathbf{a_m}$ basis as

$$\tilde{\boldsymbol{\psi}}=\boldsymbol{\psi_0}+\boldsymbol{\psi_1}\mathbf{a_1}+\boldsymbol{\psi_2}\mathbf{a_2}+\boldsymbol{\psi_3}\mathbf{a_3}, \tag{7.3}$$

then the action of $\mathbf{a_3}$ operator can be written as

$$\mathbf{a_3}\tilde{\boldsymbol{\psi}}=\boldsymbol{\psi_3}-i\boldsymbol{\psi_2}\mathbf{a_1}+i\boldsymbol{\psi_1}\mathbf{a_2}+\boldsymbol{\psi_0}\mathbf{a_3}. \tag{7.4}$$

Let us consider the eigenfunctions of $\mathbf{a_3}$ operator. The equation for the eigenvalues and eigenfunctions in this case has the following form:

$$\mathbf{a_3}\tilde{\boldsymbol{\psi}}=\lambda\tilde{\boldsymbol{\psi}}.$$

Performing sedeonic multiplication we get this equation in expanded form:

$$\boldsymbol{\psi_3}-i\boldsymbol{\psi_2}\mathbf{a_1}+i\boldsymbol{\psi_1}\mathbf{a_2}+\boldsymbol{\psi_0}\mathbf{a_3}=\lambda(\boldsymbol{\psi_0}+\boldsymbol{\psi_1}\mathbf{a_1}+\boldsymbol{\psi_2}\mathbf{a_2}+\boldsymbol{\psi_3}\mathbf{a_3}). \tag{7.5}$$

This equation is equivalent to the following system:

$$\begin{aligned}\boldsymbol{\psi_3}&=\lambda\boldsymbol{\psi_0},\\-i\boldsymbol{\psi_2}&=\lambda\boldsymbol{\psi_1},\\i\boldsymbol{\psi_1}&=\lambda\boldsymbol{\psi_2},\\\boldsymbol{\psi_0}&=\lambda\boldsymbol{\psi_3},\end{aligned} \tag{7.6}$$

from which follows

$$\begin{aligned}&\lambda^2 = 1,\\ &\boldsymbol{\psi}_0 = \lambda \boldsymbol{\psi}_3,\\ &\boldsymbol{\psi}_2 = i\lambda \boldsymbol{\psi}_1.\end{aligned} \tag{7.7}$$

Thus, the eigenvalues of $\mathbf{a}_3$ operator are equal $\lambda = \pm 1$ and eigenfunctions of $\mathbf{a}_3$ operator can be written as

$$\tilde{\boldsymbol{\psi}} = (1 + \lambda \mathbf{a}_3)\boldsymbol{\psi}_0 + (\mathbf{a}_1 + \lambda i \mathbf{a}_2)\boldsymbol{\psi}_1 , \tag{7.8}$$

where $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_1$ are the arbitrary sedeon-scalars. So, we can choose the set of functions

$$\begin{aligned}&(1 + \lambda \mathbf{a}_3),\\ &(\mathbf{a}_1 + \lambda i \mathbf{a}_2),\end{aligned} \tag{7.9}$$

as the new basis. The expressions (7.9) are the eigenfunctions of $\mathbf{a}_3$ operator.

To describe a particle in an external gravito-electromagnetic field the following change in quantum mechanical operators in equations should be made:

$$\begin{aligned}&\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{i}{\hbar}\left(q_e \varphi_e - q_g \varphi_g\right),\\ &\vec{\nabla} \to \vec{\nabla} - \frac{i}{\hbar c}\left(q_e \vec{A}_e - q_g \vec{A}_g\right).\end{aligned} \tag{7.10}$$

It leads us to the following wave equation:

$$\begin{aligned}&\left\{ i\mathbf{e}_t \frac{1}{c}\left(\frac{\partial}{\partial t} + \frac{i}{\hbar}\left(q_e\varphi_e - q_g\varphi_g\right)\right) - \mathbf{e}_r\left(\vec{\nabla} - \frac{i}{\hbar c}\left(q_e\vec{A}_e - q_g\vec{A}_g\right)\right) - i\mathbf{e}_{tr}\frac{m_0 c}{\hbar}\right\}\\ &\times\left\{ i\mathbf{e}_t \frac{1}{c}\left(\frac{\partial}{\partial t} + \frac{i}{\hbar}\left(q_e\varphi_e - q_g\varphi_g\right)\right) - \mathbf{e}_r\left(\vec{\nabla} - \frac{i}{\hbar c}\left(q_e\vec{A}_e - q_g\vec{A}_g\right)\right) - i\mathbf{e}_{tr}\frac{m_0 c}{\hbar}\right\}\tilde{\boldsymbol{\psi}} = 0.\end{aligned} \tag{7.11}$$

In the next section we will consider the task about charged relativistic particle in homogeneous magnetic field.

7.2. Relativistic particle in homogeneous magnetic field

Let us consider the relativistic particle with electrical charge q_e in an external homogeneous magnetic field directed along Z axis:

$$\vec{H}_e = H_e \mathbf{a_3} . \tag{7.12}$$

Here H_e is module of vector $\vec{H}_e$. Let us choose the vector potential $\vec{A}_e$ satisfying the gauge condition $\left(\vec{\nabla} \cdot \vec{A}_e\right) = 0$ in Landau presentation:

$$\vec{A}_e = A_{e2} \mathbf{a_2} = H_e x \mathbf{a_2} . \tag{7.13}$$

Then the sedeonic equation for relativistic particle (7.11) is written as:

$$\left\{ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta + \frac{m_0^2 c^2}{\hbar^2} + \frac{2i}{\hbar c} q_e H_e x \frac{\partial}{\partial y} + \frac{q_e^{\,2} H_e^{\,2}}{\hbar^2 c^2} x^2 - \frac{q_e \vec{H}_e}{\hbar c} \right\} \tilde{\psi} = 0 . \tag{7.14}$$

In (7.14) last term $q_e \vec{H}_e / \hbar c$ is the vector operator, which transforms the sedeonic basis of wave function. For stationary states with energy E we get:

$$\left[-\Delta + \frac{2i}{\hbar c} q_e H_e x \frac{\partial}{\partial y} + \frac{m_0^2 c^2}{\hbar^2} + \frac{q_e^{\,2} H_e^{\,2}}{\hbar^2 c^2} x^2 - \frac{q_e H_e}{\hbar c} \mathbf{a_3} \right] \tilde{\psi} = \frac{E^2}{\hbar^2 c^2} \tilde{\psi} . \tag{7.15}$$

This equation can be considered as the equation on the eigenvalues and eigenfunctions of complicated operator written in square brackets in the left part. Since this operator commutes with operators $\hat{p}_y$ and $\hat{p}_z$, all these operators have the same system of eigenfunctions. Therefore we will find the solution of (7.15) in the form

$$\tilde{\psi} = \tilde{\varphi}(x) \exp\left\{ \frac{i}{\hbar} \left(p_y y + p_z z \right) \right\} , \tag{7.16}$$

where p_y and p_z are the mouton integrals and $\tilde{\varphi}(x)$ is arbitrary function. Substituting (7.16) into (7.15) we get

$$\left[\frac{p_y^2}{\hbar^2} + \frac{p_z^2}{\hbar^2} + \frac{m_0^2 c^2}{\hbar^2} - \frac{\partial^2}{\partial x^2} - \frac{2 p_y}{\hbar^2 c} q_e H_e x + \frac{q_e^{\,2} H_e^{\,2}}{\hbar^2 c^2} x^2 - \frac{q_e H_e}{\hbar c} \mathbf{a_3} \right] \tilde{\varphi} = \frac{E^2}{\hbar^2 c^2} \tilde{\varphi} . \tag{7.17}$$

Note that the operator in the left part of (7.17) commute with operator $\mathbf{a_3}$, so we can find the solution as the linear combination of the eigenfunctions of the operator $\mathbf{a_3}$ (see (7.9)):

$$\tilde{\boldsymbol{\varphi}} = \left(1+\lambda \mathbf{a_3}\right)\mathbf{F}_1^{(\lambda)}(x) + \left(\mathbf{a_1} + \lambda i \mathbf{a_2}\right)\mathbf{F}_2^{(\lambda)}(x), \qquad (7.18)$$

where $\mathbf{F}_\gamma^{(\lambda)}(x)$ ($\gamma = 1,2$) are arbitrary sedeon-scalar functions. Then operator in the left part of (7.17) is scalar and this equation has the following form:

$$\left\{\frac{p_y^2}{\hbar^2} + \frac{p_z^2}{\hbar^2} + \frac{m_0^2 c^2}{\hbar^2} - \frac{\partial^2}{\partial x^2} - \frac{2p_y}{\hbar^2 c} q_e H_e x + \frac{q_e^2 H_e^2}{\hbar^2 c^2} x^2 - \lambda \frac{q_e H_e}{\hbar c}\right\}\mathbf{F}_\gamma^{(\lambda)} = \frac{E^2}{\hbar^2 c^2}\mathbf{F}_\gamma^{(\lambda)}. \qquad (7.19)$$

After algebraic transformations (7.19) can be rewrite as follows

$$\frac{\partial^2 \mathbf{F}_\gamma^{(\lambda)}}{\partial x^2} + \left[\left(\frac{E^2}{\hbar^2 c^2} - \frac{p_z^2}{\hbar^2} - \frac{m_0^2 c^2}{\hbar^2} + \lambda \frac{q_e H_e}{\hbar c}\right) - \left(\frac{q_e H_e}{\hbar c}\right)^2 \left(x - \frac{c p_y}{q_e H_e}\right)^2\right]\mathbf{F}_\gamma^{(\lambda)} = 0. \qquad (7.20)$$

This is the equation of linear oscillator [41]. The energy spectrum is defined by the following expression:

$$E_{n,\lambda}^2 = m_0^2 c^4 + p_z^2 c^2 + |q_e| H_e \hbar c (2n+1) - \lambda q_e H_e \hbar c. \qquad (7.21)$$

This set of energies is absolutely identical to the energy spectrum of particle with spin 1/2 obtained from the relativistic second-order equation following from the spinor Dirac equation [41].

7.3. Relativistic first-order wave equation

Let us consider the special case of particles, which are described by sedeonic first-order wave equation

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\tilde{\boldsymbol{\psi}} = 0. \qquad (7.22)$$

This equation has the solution in the form of plane wave with frequency ω and wave vector $\vec{k}$. In this case the dependence of the frequency on the wave vector has two branches:

$$\omega_\pm = \pm\sqrt{c^2 \vec{k}^2 + \frac{m_0^2 c^4}{\hbar^2}}. \qquad (7.23)$$

The plane wave solution can be written in the following generalized sedeonic form:

$$\tilde{\boldsymbol{\psi}} = \left(\mathbf{e}_1 \frac{\omega_{\pm}}{c} - i\mathbf{e}_2 \vec{k} - i\mathbf{e}_3 \frac{m_0 c}{\hbar} \right) \tilde{\boldsymbol{\varphi}} \exp\left\{ -i\omega_{\pm} t + i\left(\vec{k} \cdot \vec{r} \right) \right\}, \tag{7.24}$$

where $\tilde{\boldsymbol{\varphi}}$ is arbitrary sedeon with constant components, which do not depend from coordinates and time. Substituting (7.24) into (7.23) one can see that in this case wave equation contain algebraic zero divisor:

$$\left(\mathbf{e}_1 \frac{\omega_{\pm}}{c} - i\mathbf{e}_2 \vec{k} - i\mathbf{e}_3 \frac{m_0 c}{\hbar} \right)\left(\mathbf{e}_1 \frac{\omega_{\pm}}{c} - i\mathbf{e}_2 \vec{k} - i\mathbf{e}_3 \frac{m_0 c}{\hbar} \right) \equiv 0 . \tag{7.25}$$

The details of a plane wave solution are discussed in Section 8.6.

7.4. Conclusion

We can make some general statements about the form of the wave function of a particle in a stationary state corresponding to the certain eigenvalues of the operator $\mathbf{a}_3$. In the stationary state with energy E the wave function can be represented in the following form:

$$\tilde{\boldsymbol{\psi}}(\vec{r},t) = \tilde{\boldsymbol{\psi}}(\vec{r}) e^{-i\omega t} , \tag{7.26}$$

where frequency $\omega = E/\hbar$. For the states corresponding to certain eigenvalues of the operator $\mathbf{a}_3$ the spatial part of the wave function can be written in the form (7.18) as

$$\tilde{\boldsymbol{\psi}}_{\lambda}(\vec{r},t) = \left\{ \left(1 + \lambda \mathbf{a}_3\right) \mathbf{F}_1^{(\lambda)}(\vec{r}) + \left(\mathbf{a}_1 + \lambda i \mathbf{a}_2 \right) \mathbf{F}_2^{(\lambda)}(\vec{r}) \right\} e^{-i\omega t} . \tag{7.27}$$

This function has quite clear geometrical structure. The real and imaginary parts of the component $\left(1 + \lambda \mathbf{a}_3\right) e^{-i\omega t}$ are the combinations of an absolute vector directed parallel to the Z axis and an absolute scalar oscillating with the frequency ω. Here the phase difference between oscillations of scalar and vector parts equals 0 in case $\lambda = 1$ or π in case $\lambda = -1$. On the other hand, the real and imaginary parts of the component $\left(\boldsymbol{a}_1 + \lambda i \boldsymbol{a}_2 \right) e^{-i\omega t}$ have the form of absolute vector rotating in plane perpendicular to the Z axis with the frequency ω. The direction of rotation depends on the sign of λ.

The space-time structure of the wave function is defined by particular form of the scalar functions $\mathbf{F}_1^{(\lambda)}(\vec{r})$ and $\mathbf{F}_2^{(\lambda)}(\vec{r})$.

Thus it is shown that the sedeonic wave function of a particle in the state with defined spin projection has the specific space-time structure in the form of a sedeonic oscillator with two spatial polarizations: longitudinal linear and transverse circular.

Chapter 8. Massive fields

The attempts to generalize the second-order wave equation for massive fields on the basis of different systems of hypercomplex numbers such as quaternions and octonions have been made in Refs. [15, 42-45]. The authors discussed the possibility of constructing the field equations similar to the equations of electrodynamics but with a massive ”photon”. In particular they tried to represent the wave equation as the system of first-order Maxwell-like equations. The resulting Proca-Maxwell equations enclose field’s strengths and potentials [15,44]. On the other hand, there are a few studies concerning the generalization of the Dirac wave equation on the basis of hypercomplex numbers [22,46-49]. In this approach, the wave function has a scalar-vector structure similar in nature with the potential of field and the hypercomplex Dirac-like equation can be reformulated as the wave equation for the potential of special field.

The consideration of multicomponent wave functions is an inevitable necessity in describing the spin and space-time properties of fields and quantum systems. The requirements of relativistic invariance leads to the necessity of introducing sixteen-component algebras taking into account the full symmetry with respect to the spatial and time inversion. There are a few approaches in the development of field theory on the basis of sixteen-component structures. One of them is the application of hypernumbers sedenions, which are obtained from octonions by Cayley-Dickson extension procedure [4,50]. However the essential imperfection of sedenions is their nonassociativity. Another approach is based on the application of hypercomplex multivectors generating associative space-time Clifford algebras [14]. The basic idea of such multivectors is an introduction of additional noncommutative time unit vector, which is orthogonal to the space unit vectors. However, the application of such multivectors in quantum mechanics and field theory is considered in general as one of abstract algebraic scheme enabling the reformulation of Klein-Gordon and Dirac equations for the multicomponent wave functions but does not touch the physical entity of these equations.

In this section we consider the massive fields described by first- and second-order wave equations on the basis of sedeonic potentials and space-time operators [19].

8.1. Generalized sedeonic equation for baryon field

Let us consider the sedeonic wave equation for free massive field:

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2 \vec{\nabla} - i\mathbf{e}_3 \frac{m_0 c}{\hbar} \right)\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2 \vec{\nabla} - i\mathbf{e}_3 \frac{m_0 c}{\hbar} \right)\tilde{\mathbf{W}} = 0 , \qquad (8.1)$$

where m_0 is the mass of quantum of massive field and $\tilde{\mathbf{W}}$ is sedeonic potential. For convenience we will write:

$$\partial = \frac{1}{c}\frac{\partial}{\partial t}, \qquad m = \frac{m_0 c}{\hbar}. \qquad (8.2)$$

Then we can rewrite equation (8.1) in compact form:

$$\left(i\mathbf{e}_1 \partial - \mathbf{e}_2 \vec{\nabla} - i\mathbf{e}_3 m \right)\left(i\mathbf{e}_1 \partial - \mathbf{e}_2 \vec{\nabla} - i\mathbf{e}_3 m \right)\tilde{\mathbf{W}} = 0. \qquad (8.3)$$

Let us choose the potential in the following form:

$$\tilde{\mathbf{W}} = a + i\mathbf{e}_1 b - i\mathbf{e}_2 c - i\mathbf{e}_3 d + i\vec{A} + \mathbf{e}_1 \vec{B} + \mathbf{e}_2 \vec{C} - \mathbf{e}_3 \vec{D} , \qquad (8.4)$$

where the components $a, b, c, d, \vec{A}, \vec{B}, \vec{C}, \vec{D}$ are the functions of spatial coordinates and time. Introducing the scalar and vector fields strengths according to the following definitions:

$$
\begin{aligned}
&e = \partial b + \left(\vec{\nabla}\cdot\vec{C}\right) + md,\\
&f = \partial a + \left(\vec{\nabla}\cdot\vec{D}\right) + mc,\\
&g = \partial d + \left(\vec{\nabla}\cdot\vec{A}\right) - mb,\\
&h = \partial c + \left(\vec{\nabla}\cdot\vec{B}\right) - ma,\\
&\vec{E} = -\partial\vec{B} - \vec{\nabla}c - i\left[\vec{\nabla}\times\vec{C}\right] - m\vec{D},\\
&\vec{F} = -\partial\vec{A} - \vec{\nabla}d + i\left[\vec{\nabla}\times\vec{D}\right] - m\vec{C},\\
&\vec{G} = -\partial\vec{D} - \vec{\nabla}a - i\left[\vec{\nabla}\times\vec{A}\right] + m\vec{B},\\
&\vec{H} = -\partial\vec{C} - \vec{\nabla}b + i\left[\vec{\nabla}\times\vec{B}\right] + m\vec{A},
\end{aligned}
\tag{8.5}
$$

we get

$$
\begin{aligned}
&\left(i\mathbf{e}_1\partial - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right)\left(a + i\mathbf{e}_1 b - i\mathbf{e}_2 c - i\mathbf{e}_3 d + i\vec{A} + \mathbf{e}_1\vec{B} + \mathbf{e}_2\vec{C} - \mathbf{e}_3\vec{D}\right)\\
&= -e + i\mathbf{e}_1 f - i\mathbf{e}_2 g + i\mathbf{e}_3 h - i\vec{E} + \mathbf{e}_1\vec{F} + \mathbf{e}_2\vec{G} + \mathbf{e}_3\vec{H},
\end{aligned}
\tag{8.6}
$$

and the wave equation (8.3) takes the form

$$
\left(i\mathbf{e}_1\partial - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right)\left(-e + i\mathbf{e}_1 f - i\mathbf{e}_2 g + i\mathbf{e}_3 h - i\vec{E} + \mathbf{e}_1\vec{F} + \mathbf{e}_2\vec{G} + \mathbf{e}_3\vec{H}\right) = 0\,. \tag{8.7}
$$

Performing the action of operator in the left part of the equation (8.7), and separating the terms with different space-time properties, we obtain the system of equations for the field's strengths, similar to the system of Maxwell's equations in electrodynamics:

$$
\begin{aligned}
&\partial f + \left(\vec{\nabla} \cdot \vec{G}\right) - mh = 0\,, \\
&\partial e + \left(\vec{\nabla} \cdot \vec{H}\right) - mg = 0\,, \\
&\partial h + \left(\vec{\nabla} \cdot \vec{E}\right) + mf = 0\,, \\
&\partial g + \left(\vec{\nabla} \cdot \vec{F}\right) + me = 0, \\
&\partial \vec{F} + \vec{\nabla} g + i\left[\vec{\nabla} \times \vec{G}\right] - m\vec{H} = 0\,, \\
&\partial \vec{E} + \vec{\nabla} h - i\left[\vec{\nabla} \times \vec{H}\right] - m\vec{G} = 0\,, \\
&\partial \vec{H} + \vec{\nabla} e + i\left[\vec{\nabla} \times \vec{E}\right] + m\vec{F} = 0\,, \\
&\partial \vec{G} + \vec{\nabla} f - i\left[\vec{\nabla} \times \vec{F}\right] + m\vec{E} = 0\,.
\end{aligned}
\tag{8.8}
$$

The proposed equations for massive field possess a specific gauge invariance. It is easy to see that fields strengths (8.5) and equations (8.8) are not changed under the following substitutions for potentials:

$$
\begin{aligned}
&a \Rightarrow a + \partial\varepsilon_a - m\varepsilon_c, \\
&b \Rightarrow b + \partial\varepsilon_b - m\varepsilon_d, \\
&c \Rightarrow c + \partial\varepsilon_c + m\varepsilon_a, \\
&d \Rightarrow d + \partial\varepsilon_d + m\varepsilon_b, \\
&\vec{A} \Rightarrow \vec{A} - \vec{\nabla}\varepsilon_d, \\
&\vec{B} \Rightarrow \vec{B} - \vec{\nabla}\varepsilon_c, \\
&\vec{C} \Rightarrow \vec{C} - \vec{\nabla}\varepsilon_b, \\
&\vec{D} \Rightarrow \vec{D} - \vec{\nabla}\varepsilon_a,
\end{aligned}
\tag{8.9}
$$

where ε_a, ε_b, ε_c, ε_d are arbitrary scalar functions, which satisfy homogeneous Klein-Gordon equation. These gauge conditions are different from those taken in electrodynamics.

Multiplying each of the equations (8.8) to the corresponding field strength and adding these equations to each other, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\partial\left(f^2+e^2+h^2+g^2+\vec{F}^2+\vec{E}^2+\vec{H}^2+\vec{G}^2\right)\\
&+f\left(\vec{\nabla}\cdot\vec{G}\right)+e\left(\vec{\nabla}\cdot\vec{H}\right)+h\left(\vec{\nabla}\cdot\vec{E}\right)+g\left(\vec{\nabla}\cdot\vec{F}\right)\\
&+\left(\vec{F}\cdot\vec{\nabla}g\right)+\left(\vec{E}\cdot\vec{\nabla}h\right)+\left(\vec{H}\cdot\vec{\nabla}e\right)+\left(\vec{G}\cdot\vec{\nabla}f\right)\\
&+i\left(\vec{F}\cdot\left[\vec{\nabla}\times\vec{G}\right]\right)-i\left(\vec{E}\cdot\left[\vec{\nabla}\times\vec{H}\right]\right)+i\left(\vec{H}\cdot\left[\vec{\nabla}\times\vec{E}\right]\right)-i\left(\vec{G}\cdot\left[\vec{\nabla}\times\vec{F}\right]\right)=0\,.
\end{aligned}
\tag{8.10}
$$

Let us introduce the following notations:

$$
\begin{aligned}
&w=-\frac{1}{8\pi}\left(e^2+f^2+g^2+h^2+\vec{E}^2+\vec{F}^2+\vec{G}^2+\vec{H}^2\right),\\
&\vec{P}=-\frac{c}{4\pi}\left(e\vec{H}+f\vec{G}+g\vec{F}+h\vec{E}+i\left[\vec{E}\times\vec{H}\right]+i\left[\vec{G}\times\vec{F}\right]\right).
\end{aligned}
\tag{8.11}
$$

Then the equation (8.9) can be written as:

$$
\frac{\partial w}{\partial t}+\left(\vec{\nabla}\cdot\vec{P}\right)=0 \quad . \tag{8.12}
$$

This expression is an analogy to Poynting's theorem for massive fields. The w term plays the role of field energy density and $\vec{P}$ is an energy flux density vector. The minus signs in expressions (8.11) are because we assume that stationary scalar point sources of equal baryon charge attract one another (see section 8.3).

8.2. Nonhomogeneous equation of baryon field

Let us consider nonhomogeneous sedeonic equation for massive field with phenomenological source. In this case the field potential satisfy the following equation:

$$
\left(i\mathbf{e}_1\partial-\mathbf{e}_2\vec{\nabla}-i\mathbf{e}_3 m\right)\left(i\mathbf{e}_1\partial-\mathbf{e}_2\vec{\nabla}-i\mathbf{e}_3 m\right)\tilde{\mathbf{W}}=\tilde{\mathbf{J}}. \tag{8.13}
$$

In analogy to gravitodynamics we consider a four-component source sedeon

$$
\tilde{\mathbf{J}}=-i\mathbf{e}_1 4\pi\rho_B-\mathbf{e}_2\frac{4\pi}{c}\vec{j}_B\,, \tag{8.14}
$$

where ρ_B is a volume density of baryon charge and $\vec{j}_B$ is volume density of baryon current. In this case the sedeonic potential can be choosen as

$$\tilde{\mathbf{W}} = i\mathbf{e}_1 b + \mathbf{e}_2 \vec{C}\,, \tag{8.15}$$

where $b(\vec{r},t)$ is a scalar part (time component) and $\vec{C}(\vec{r},t)$ is vector part (spatial component) of four-dimensional baryon potential. Then we have only the following nonzero field's strengths:

$$\begin{aligned} e &= \partial b + \left(\vec{\nabla}\cdot\vec{C}\right), \\ g &= -mb, \\ \vec{E} &= -i\left[\vec{\nabla}\times\vec{C}\right], \\ \vec{F} &= -m\vec{C}, \\ \vec{H} &= -\partial\vec{C} - \vec{\nabla}b. \end{aligned} \tag{8.16}$$

The wave equation (8.13) takes the form

$$\begin{aligned} &\left(i\mathbf{e}_1\partial - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right)\left(-e - i\mathbf{e}_2 g - i\vec{E} + \mathbf{e}_1\vec{F} + \mathbf{e}_3\vec{H}\right) \\ &= -i\mathbf{e}_1 4\pi\rho_B - \mathbf{e}_2\frac{4\pi}{c}\vec{j}_B. \end{aligned} \tag{8.17}$$

The system of equations for the baryon field is written as

$$\begin{aligned} &\partial e + \left(\vec{\nabla}\cdot\vec{H}\right) - mg = 4\pi\rho_B\,, \\ &\left(\vec{\nabla}\cdot\vec{E}\right) = 0\,, \\ &\partial g + \left(\vec{\nabla}\cdot\vec{F}\right) + me = 0\,, \\ &\partial\vec{F} + \vec{\nabla}g - m\vec{H} = 0\,, \\ &\partial\vec{E} - i\left[\vec{\nabla}\times\vec{H}\right] = 0\,, \\ &\partial\vec{H} + \vec{\nabla}e + i\left[\vec{\nabla}\times\vec{E}\right] + m\vec{F} = -\frac{4\pi}{c}\vec{j}_B\,, \\ &-i\left[\vec{\nabla}\times\vec{F}\right] + m\vec{E} = 0\,. \end{aligned} \tag{8.18}$$

On the other hand, applying the operator $\left(i\mathbf{e}_1\partial - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right)$ to the equation (8.17), we obtain the following wave equations for the field strengths:

$$\begin{aligned}
&\left(\partial^2-\Delta+m^2\right)e=4\pi\left(\partial\rho_B+\frac{1}{c}\left(\vec{\nabla}\cdot\vec{j}_B\right)\right),\\
&\left(\partial^2-\Delta+m^2\right)g=-4\pi m\rho_B,\\
&\left(\partial^2-\Delta+m^2\right)\vec{F}=-m\frac{4\pi}{c}\vec{j}_B,\\
&\left(\partial^2-\Delta+m^2\right)\vec{E}=-i\frac{4\pi}{c}\left[\vec{\nabla}\times\vec{j}_B\right],\\
&\left(\partial^2-\Delta+m^2\right)\vec{H}=-4\pi\left(\frac{1}{c}\partial\vec{j}_B+\vec{\nabla}\rho_B\right).
\end{aligned}\tag{8.19}$$

Assuming baryon charge conservation

$$\partial\rho_B+\frac{1}{c}\left(\vec{\nabla}\cdot\vec{j}_B\right)=0\,,\tag{8.20}$$

we can choose the field strength e equal to zero. This is equivalent to the following gauge condition:

$$\partial b+\left(\vec{\nabla}\cdot\vec{C}\right)=0\,,\tag{8.21}$$

similar to the Lorentz gauge in electrodynamics.

8.3. Stationary field of point scalar source

In stationary case $\vec{j}_B=0$ and potential can be chosen as

$$\tilde{\mathbf{W}}=i\mathbf{e}_1 b\left(\vec{r}\right).\tag{8.22}$$

Then we have only two nonzero field components

$$\begin{aligned}
g&=-mb\,,\\
\vec{H}&=-\vec{\nabla}b
\end{aligned}\tag{8.23}$$

and the following field equations:

$$\left(\vec{\nabla}\cdot\vec{H}\right)-mg=4\pi\rho_B\ ,$$
$$\vec{\nabla}g-m\vec{H}=0\ , \qquad (8.24)$$
$$\left[\vec{\nabla}\times\vec{H}\right]=0\ .$$

Let us calculate the field produced by a scalar stationary point source

$$\tilde{\mathbf{J}}=-i\mathbf{e}_1 4\pi q_B\delta(\vec{r})\ , \qquad (8.25)$$

where q_B is point baryon charge. Then stationary wave equation can be written in spherical coordinates as

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{m_0^2c^2}{\hbar^2}\right)b\left(\vec{r}\right)=-\,4\pi q_B\delta\left(\vec{r}\right). \qquad (8.26)$$

The partial solution of the equation (8.26), which decays at $r\to\infty$, is

$$b=\frac{q_B}{r}\exp\left(-\frac{m_0c}{\hbar}r\right). \qquad (8.27)$$

Thus, the stationary field of baryon point source has scalar and vector components

$$g_B=-\frac{m_0c}{\hbar}\frac{q_B}{r}\exp\left(-\frac{m_0c}{\hbar}r\right), \qquad (8.28)$$

$$\vec{H}_B=\left(\frac{1}{r}+\frac{m_0c}{\hbar}\right)\frac{q_B}{r}\exp\left(-\frac{m_0c}{\hbar}r\right)\vec{r}_0\ , \qquad (8.29)$$

where $\vec{r}_0$ is a unit radial vector.

8.4. Baryon – baryon interaction

Let us consider the interaction of two point baryon charges due to the overlap of their fields. Taking into account that the field in this case is the sum of the two fields $g=g_{B1}+g_{B2}$ and $\vec{H}=\vec{H}_{B1}+\vec{H}_{B2}$ the energy of interaction (see expression (8.11)) is equal

$$W_{BB}=-\frac{1}{4\pi}\int\left\{g_{B1}g_{B2}+\left(\vec{H}_{B1}\cdot\vec{H}_{B2}\right)\right\}dV\ , \qquad (8.30)$$

where the integral is over all space. This expression can be derived analytically. Substituting (8.28) and (8.29) we obtain

$$W_{BB} = -\frac{q_{B1}q_{B2}}{R}\exp\left(-\frac{m_0 c}{\hbar}R\right), \tag{8.31}$$

where R is the distance between point baryons. By definition we assume interaction between equal charges to be attractive.

8.5. Sedeonic equation for lepton field

In [22] we supposed that lepton fields can be described by sedeonic first-order equation, similar to the Dirac equation. In sedeonic algebra the homogeneous first-order equation is written as

$$\left(i\mathbf{e}_1\partial - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right)\tilde{\mathbf{W}} = 0. \tag{8.32}$$

In equation (8.32) the basis elements $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ play the role of space-time operators, which transform the wave function by means of component permutation. Choosing potential $\tilde{\mathbf{W}}$ in the form (8.4) we find that sedeonic equation (8.32) is equivalent to the following system

$$\begin{gathered}
\partial a + \left(\vec{\nabla}\cdot\vec{D}\right) + mc = 0,\\
\partial b + \left(\vec{\nabla}\cdot\vec{C}\right) + md = 0,\\
\partial c + \left(\vec{\nabla}\cdot\vec{B}\right) - ma = 0,\\
\partial d + \left(\vec{\nabla}\cdot\vec{A}\right) - mb = 0,\\
\partial\vec{A} + \vec{\nabla}d - i\left[\vec{\nabla}\times\vec{D}\right] + m\vec{C} = 0,\\
\partial\vec{B} + \vec{\nabla}c + i\left[\vec{\nabla}\times\vec{C}\right] + m\vec{D} = 0,\\
\partial\vec{C} + \vec{\nabla}b - i\left[\vec{\nabla}\times\vec{B}\right] - m\vec{A} = 0,\\
\partial\vec{D} + \vec{\nabla}a + i\left[\vec{\nabla}\times\vec{A}\right] - m\vec{B} = 0.
\end{gathered} \tag{8.33}$$

In fact these equations describe the special fields with zero field strengths [22] (see expression (8.5)).

Multiplying each equation of system (8.33) on corresponding components of potential $\tilde{\mathbf{W}}$ and adding we get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(a^2+b^2+c^2+d^2+\vec{A}^2+\vec{B}^2+\vec{C}^2+\vec{D}^2\right)\\
&+a\left(\vec{\nabla}\cdot\vec{D}\right)+b\left(\vec{\nabla}\cdot\vec{C}\right)+c\left(\vec{\nabla}\cdot\vec{B}\right)+d\left(\vec{\nabla}\cdot\vec{A}\right)\\
&+\left(\vec{A}\cdot\vec{\nabla}d\right)+\left(\vec{B}\cdot\vec{\nabla}c\right)+\left(\vec{C}\cdot\vec{\nabla}b\right)+\left(\vec{D}\cdot\vec{\nabla}a\right)\\
&-i\left(\vec{A}\cdot\left[\vec{\nabla}\times\vec{D}\right]\right)+i\left(\vec{B}\cdot\left[\vec{\nabla}\times\vec{C}\right]\right)\\
&-i\left(\vec{C}\cdot\left[\vec{\nabla}\times\vec{B}\right]\right)+i\left(\vec{D}\cdot\left[\vec{\nabla}\times\vec{A}\right]\right)=0.
\end{aligned}\tag{8.34}$$

Let us introduce the following notations:

$$\mathrm{W}=\frac{1}{8\pi}\left(a^2+b^2+c^2+d^2+\vec{A}^2+\vec{B}^2+\vec{C}^2+\vec{D}^2\right),\tag{8.35}$$

$$\vec{S}=\frac{c}{4\pi}\left(a\vec{D}+b\vec{C}+c\vec{B}+d\vec{A}+i\left[\vec{A}\times\vec{D}\right]+i\left[\vec{C}\times\vec{B}\right]\right).\tag{8.36}$$

Then the equation (8.34) can be represented as

$$\frac{\partial \mathrm{W}}{\partial t}+\left(\vec{\nabla}\cdot\vec{S}\right)=0\quad .\tag{8.37}$$

This expression is an analogue of the Poynting theorem for massive fields (described by first-order equation) but written for field potentials.

8.6. Plane wave solution of first-order equation

The homogeneous first-order wave equation

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}-i\mathbf{e}_3\frac{m_0 c}{\hbar}\right)\tilde{\mathbf{W}}=0\tag{8.38}$$

has the solution in the form of plane wave. In this case the potential can be written as

$$\tilde{\mathbf{W}}=\tilde{\mathbf{U}}\exp\left\{-i\omega t+i\left(\vec{k}\cdot\vec{r}\right)\right\},\tag{8.39}$$

where ω is frequency and $\vec{k}$ is an absolute wave vector. The wave amplitude $\tilde{\mathbf{U}}$ does not depend on the coordinates and time. In this case, the dependence of frequency on the wave vector has two branches:

$$\omega_{\pm} = \pm\sqrt{c^2k^2 + \frac{m_0^2c^4}{\hbar^2}} \ . \tag{8.40}$$

Substituting (8.39) in equation (8.38) and taking into account (8.40), we obtain

$$\left(\mathbf{e}_1 \frac{\omega_{\pm}}{c} - i\mathbf{e}_2\vec{k} - i\mathbf{e}_3 \frac{m_0 c}{\hbar}\right)\tilde{\mathbf{U}} = 0 \ . \tag{8.41}$$

For convenience we introduce the following notations:

$$\omega' = \frac{\omega_{\pm}}{c} \ ,$$

$$m = \frac{m_0 c}{\hbar} \ .$$

Then the equation (8.41) is written as

$$\left(\mathbf{e}_1\omega' - i\mathbf{e}_2\vec{k} - i\mathbf{e}_3 m\right)\tilde{\mathbf{U}} = 0 \ . \tag{8.42}$$

Let us consider the amplitude of the wave function in the form of (8.4):

$$\tilde{\mathbf{U}} = a + i\mathbf{e}_1 b - i\mathbf{e}_2 c - i\mathbf{e}_3 d + i\vec{A} + \mathbf{e}_1\vec{B} + \mathbf{e}_2\vec{C} - \mathbf{e}_3\vec{D} \ ,$$

where a, b, c, d are arbitrary constants and $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ are arbitrary vectors. Then the equation (8.42) takes the form

$$\begin{aligned} &\left(\mathbf{e}_1\omega' - i\mathbf{e}_2\vec{k} - i\mathbf{e}_3 m\right) \\ &\times\left(a + i\mathbf{e}_1 b - i\mathbf{e}_2 c - i\mathbf{e}_3 d + i\vec{A} + \mathbf{e}_1\vec{B} + \mathbf{e}_2\vec{C} - \mathbf{e}_3\vec{D}\right) = 0. \end{aligned} \tag{8.43}$$

Let us represent the vector constants as parallel and perpendicular to fixed wave vector $\vec{k}$

$$\begin{aligned}\vec{A} &= \vec{A}_{\parallel} + \vec{A}_{\perp},\\ \vec{B} &= \vec{B}_{\parallel} + \vec{B}_{\perp},\\ \vec{C} &= \vec{C}_{\parallel} + \vec{C}_{\perp},\\ \vec{D} &= \vec{D}_{\parallel} + \vec{D}_{\perp}.\end{aligned} \tag{8.44}$$

Then performing the multiplication in (8.43), we obtain the following system of algebraic equations:

$$\begin{aligned}&\omega' b - kC_{\parallel} + imd = 0,\\ &\omega' a - kD_{\parallel} + imc = 0,\\ &\omega' d - kA_{\parallel} - imb = 0,\\ &\omega' c - kB_{\parallel} - ima = 0,\end{aligned} \tag{8.45}$$

$$\begin{aligned}&\omega' B_{\parallel} - kc + imD_{\parallel} = 0,\\ &\omega' A_{\parallel} - kd + imC_{\parallel} = 0,\\ &\omega' D_{\parallel} - ka - imB_{\parallel} = 0,\\ &\omega' C_{\parallel} - kb - imA_{\parallel} = 0,\end{aligned} \tag{8.46}$$

$$\begin{aligned}&\omega' \vec{B}_{\perp} - i\left[\vec{k} \times \vec{C}_{\perp}\right] + im\vec{D}_{\perp} = 0,\\ &i\omega' \vec{A}_{\perp} - \left[\vec{k} \times \vec{D}_{\perp}\right] - m\vec{C}_{\perp} = 0,\\ &i\omega' \vec{D}_{\perp} + \left[\vec{k} \times \vec{A}_{\perp}\right] + m\vec{B}_{\perp} = 0,\\ &i\omega' \vec{C}_{\perp} - \left[\vec{k} \times \vec{B}_{\perp}\right] + m\vec{A}_{\perp} = 0,\end{aligned} \tag{8.47}$$

where $A_{\parallel}$, $B_{\parallel}$, $C_{\parallel}$ and $D_{\parallel}$ are projections of corresponding vectors on $\vec{k}$ direction. From equations (8.45) we get following relations:

$$\begin{aligned}A_{\parallel} &= \frac{\omega'}{k} d - i\frac{m}{k} b,\\ B_{\parallel} &= \frac{\omega'}{k} c - i\frac{m}{k} a,\\ C_{\parallel} &= \frac{\omega'}{k} b + i\frac{m}{k} d,\\ D_{\parallel} &= \frac{\omega'}{k} a + i\frac{m}{k} c.\end{aligned} \tag{8.48}$$

On the other hand, from equation (8.47) we obtain

$$\vec{C}_{\perp} = \frac{im}{\omega'}\vec{A}_{\perp} - \frac{i}{\omega'}\left[\vec{k}\times\vec{B}_{\perp}\right],$$
$$\vec{D}_{\perp} = \frac{im}{\omega'}\vec{B}_{\perp} + \frac{i}{\omega'}\left[\vec{k}\times\vec{A}_{\perp}\right]. \qquad (8.49)$$

Then the amplitude $\tilde{\mathbf{U}}$ can be written as

$$\begin{aligned}\tilde{\mathbf{U}} = {} & a + i\mathbf{e}_1 b - i\mathbf{e}_2 c - i\mathbf{e}_3 d \\ & + \left\{ i\omega' d + mb + \mathbf{e}_1\omega' c - i\mathbf{e}_1 ma \right\}\frac{\vec{k}}{k^2} \\ & + \left\{ \mathbf{e}_2\omega' b + i\mathbf{e}_2 md - \mathbf{e}_3\omega' a - i\mathbf{e}_3 mc \right\}\frac{\vec{k}}{k^2} \\ & + i\vec{A}_{\perp} + \mathbf{e}_1\vec{B}_{\perp} + i\mathbf{e}_2\frac{m}{\omega'}\vec{A}_{\perp} - i\mathbf{e}_3\frac{m}{\omega'}\vec{B}_{\perp} \\ & - i\mathbf{e}_3\frac{1}{\omega'}\left[\vec{k}\times\vec{A}_{\perp}\right] - i\mathbf{e}_2\frac{1}{\omega'}\left[\vec{k}\times\vec{B}_{\perp}\right].\end{aligned} \qquad (8.50)$$

The expression (8.50) can be represented in more compact form

$$\tilde{\mathbf{U}} = \left(\mathbf{e}_1\omega' - i\mathbf{e}_2\vec{k} - i\mathbf{e}_3 m\right)\left\{ i\mathbf{e}_2\frac{\vec{k}}{k^2}\left(a + i\mathbf{e}_1 b - i\mathbf{e}_2 c - i\mathbf{e}_3 d\right) + i\mathbf{e}_1\frac{1}{\omega'}\vec{A}_{\perp} + \frac{1}{\omega'}\vec{B}_{\perp} \right\}. \qquad (8.51)$$

Substituting (8.51) in equation (8.42) and taking into account (8.40) one can see that this equation is satisfied for any parameters a, b, c, d , $\vec{A}_{\perp}, \vec{B}_{\perp}$, since we have

$$\left(\mathbf{e}_1\omega' - i\mathbf{e}_2\vec{k} - i\mathbf{e}_3 m\right)\left(\mathbf{e}_1\omega' - i\mathbf{e}_2\vec{k} - i\mathbf{e}_3 m\right) \equiv 0\,. \qquad (8.52)$$

So, the solution (8.51) contains the algebraic zero divisor.

In general case the solution of equation (8.32) can be written in the form of generalized plane wave:

$$\tilde{\mathbf{W}} = \left(\mathbf{e}_1\frac{\omega_{\pm}}{c} - i\mathbf{e}_2\vec{k} - i\mathbf{e}_3\frac{m_0 c}{\hbar} \right)\tilde{\mathbf{M}}\exp\left\{ -i\omega_{\pm}t + i\left(\vec{k}\cdot\vec{r}\right) \right\}, \qquad (8.53)$$

where $\tilde{\mathbf{M}}$ is arbitrary sedeon with constant components, which do not depend on time and coordinates.

8.7. Nonhomogeneous equation of lepton field

Let us consider the nonhomogeneous equation corresponding to the equation (8.32):

$$\left(i\mathbf{e}_1\partial - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right)\tilde{\mathbf{W}} = \tilde{\mathbf{I}}. \qquad (8.54)$$

Here $\tilde{\mathbf{I}}$ is sedeonic field source describing lepton charges and currents. Choosing the potential $\tilde{\mathbf{W}}$ in the form (8.4), we obtain following equation for the lepton field strengths:

$$-e + i\mathbf{e}_1 f - i\mathbf{e}_2 g + i\mathbf{e}_3 h - i\vec{E} + \mathbf{e}_1\vec{F} + \mathbf{e}_2\vec{G} + \mathbf{e}_3\vec{H} = \mathbf{I}_0 + \vec{\mathbf{I}}\,. \qquad (8.55)$$

This equation means that the strengths of this field are nonzero only in the region of field source.

Let us consider the sedeonic source in the following form:

$$\tilde{\mathbf{I}} = -i\mathbf{e}_2\rho_L + \mathbf{e}_1\frac{1}{c}\vec{j}_L\,, \qquad (8.56)$$

where ρ_L is a volume density of lepton charge and $\vec{j}_L$ is volume – density of lepton current. In this case the equation (8.55) is rewritten as

$$-i\mathbf{e}_2 g + \mathbf{e}_1\vec{F} = -i\mathbf{e}_2 4\pi\rho_L + \mathbf{e}_1\frac{4\pi}{c}\vec{j}_L\,. \qquad (8.57)$$

Applying the operator $\left(i\mathbf{e}_1\partial - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right)$ to the equation (8.57) and separating the values with different space-time properties we obtain the following equations for the lepton field strengths:

$$\begin{gathered}
g = 4\pi\rho_L, \\
\vec{F} = \frac{4\pi}{c}\vec{j}_L, \\
\partial g + \left(\vec{\nabla}\cdot\vec{F}\right) = 4\pi\left\{\partial\rho_L + \frac{1}{c}\left(\vec{\nabla}\cdot\vec{j}_L\right)\right\}, \\
\left[\vec{\nabla}\times\vec{F}\right] = \frac{4\pi}{c}\left[\vec{\nabla}\times\vec{j}_L\right], \\
\partial\vec{F} + \vec{\nabla}g = 4\pi\left\{\frac{1}{c}\partial\vec{j}_L + \vec{\nabla}\rho_L\right\}.
\end{gathered} \qquad (8.58)$$

Assuming lepton charge conservation

$$\partial\rho_L + \frac{1}{c}\left(\vec{\nabla}\cdot\vec{j}_L\right) = 0\,, \tag{8.59}$$

we have the following gauge condition:

$$\partial g + \left(\vec{\nabla}\cdot\vec{F}\right) = 0\,, \tag{8.60}$$

which is similar to conventional Lorentz gauge, but for field strength here.

Let us consider the a stationary lepton field generated by a scalar point source

$$\tilde{\mathbf{I}} = -i\mathbf{e_2}4\pi q_L\delta(\vec{r})\,, \tag{8.61}$$

where q_L is the point lepton charge. Then the strength of the scalar field is

$$g_L\left(\vec{r}\right) = 4\pi q_L\delta\left(\vec{r}\right). \tag{8.62}$$

This field is non-zero only in the region of source. It indicates that two point lepton charges interact only if they are at the same point of space. The interaction energy for two point charges q_{L1} and q_{L2} is equal

$$W_{LL} = -\frac{1}{4\pi}\int_V g_{L1}g_{L2}\,dV = -4\pi q_{L1}q_{L2}\delta\left(\vec{R}\right), \tag{8.63}$$

where $\vec{R}$ is the distance between point leptons.

8.8. Baryon – lepton interaction

One could suppose an interaction between baryon and lepton charges due to overlap of the scalar fields g_B and g_L. The respective fields are determined by equations (8.28) and (8.68), so that the interaction energy is equal to:

$$W_{BL} = -\frac{1}{4\pi}\int\limits_V g_B g_L dV \,. \tag{8.64}$$

As a result, we get

$$W_{BL} = -\frac{m_0 c}{\hbar}\frac{q_B q_L}{R}\exp\left(-\frac{m_0 c}{\hbar}R\right), \tag{8.65}$$

where R is the distance between point baryon and lepton.

8.9. Conclusion

Thus, we considered the sedeonic generalization of equations describing the massive field. It is shown that this approach allows to build a massive field theory analogous to the theory of massless electromagnetic field in classical electrodynamics.

We have considered the sedeonic second order wave equation for sedeon wave function. It was shown that this equation can be interpreted as the equation for the baryon field potentials. We have demonstrated that the second-order wave equation for the potentials can be represented as a system of first order equations for the field strengths similar to the system of Maxwell's equations. We generalized the concepts of energy density and energy flux for massive fields, and derive relations for the field energy and momentum similar to Poynting's theorem in electrodynamics. It was shown that in the particular case of a stationary point source the solution of the sedeonic wave equation is a potential of Yukawa-type. The energy of interaction of two point baryons is derived.

Assuming that lepton field is described by first-order wave equation, it was shown that the strengths of the lepton fields are nonzero only in the area of sources, so the point leptons interact only when they are in the same point of space. The plane wave solution of sedeonic first-order wave equation is derived.

We demonstrated the possibility of describing the baryon-lepton interaction in terms of scalar fields overlapping.

Chapter 9. Neutrino field

9.1. Sedeonic equations of neutrino field

Among the solutions of the homogeneous sedeonic wave equation of gravitoelectromagnetic field there is a special class that satisfies the sedeonic first-order equation of the following form [18]:

$$\left(i\mathbf{e}_\mathbf{t} \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_\mathbf{r}\vec{\nabla} \right)\tilde{\mathbf{W}}_\nu = 0 . \tag{9.1}$$

The field satisfying this equation will be called the neutrino field. Based on analogy with gravitoelectromagnetism (see (6.12)), we consider the potential $\tilde{\mathbf{W}}_\nu$ in the following form:

$$\tilde{\mathbf{W}}_\nu = i\mathbf{e}_\mathbf{t}\varphi_\nu + \mathbf{e}_\mathbf{r}\vec{A}_\nu , \tag{9.2}$$

where φ_ν and $\vec{A}_\nu$ are complex scalar and vector potentials of neutrino field:

$$\varphi_\nu = \varphi_e + i\varphi_g , \tag{9.3}$$

$$\vec{A}_\nu = \vec{A}_e + i\vec{A}_g . \tag{9.4}$$

Thus, the equation for free neutrino field can be written as

$$\left(i\mathbf{e}_\mathbf{t} \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_\mathbf{r}\vec{\nabla} \right)\left(i\mathbf{e}_\mathbf{t}\varphi_\nu + \mathbf{e}_\mathbf{r}\vec{A}_\nu \right) = 0 . \tag{9.5}$$

Appling the operator

$$i\mathbf{e}_\mathbf{t} \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_\mathbf{r}\vec{\nabla}$$

to the equation (9.5), we have

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \right)\left(i\mathbf{e}_\mathrm{t}\varphi_\nu + \mathbf{e}_\mathrm{r}\vec{A}_\nu \right) = 0 . \tag{9.6}$$

Separating the values with different space-time and charge properties we obtain the wave equations for the potentials

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\varphi_e=0, \tag{9.7}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\varphi_g=0, \tag{9.8}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{A}_e=0, \tag{9.9}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{A}_g=0. \tag{9.10}$$

It indicates that the potentials of neutrino field φ_e, φ_g, $\vec{A}_e$, $\vec{A}_g$ satisfy the same second-order equations as well as potentials of gravitoelectromagnetic field, however the equation (9.5) allocates only those solutions that have zero strengths of electric (and gravitoelectric) and magnetic (and gravitomagnetic) fields. Indeed, performing the sedeonic multiplication in (9.5) we have

$$-\frac{1}{c}\frac{\partial\varphi_\nu}{\partial t}-\mathbf{e}_{\mathbf{tr}}\frac{1}{c}\frac{\partial\vec{A}_\nu}{\partial t}-\mathbf{e}_{\mathbf{tr}}\vec{\nabla}\varphi_\nu-\left(\vec{\nabla}\cdot\vec{A}_\nu\right)-\left[\vec{\nabla}\times\vec{A}_\nu\right]=0. \tag{9.11}$$

Separating in (9.11) the values with different space-time and charge properties we obtain the system of equations for the potentials:

$$\begin{aligned}
&\frac{1}{c}\frac{\partial\varphi_e}{\partial t}+\left(\vec{\nabla}\cdot\vec{A}_e\right)=0,\\
&\frac{1}{c}\frac{\partial\vec{A}_e}{\partial t}+\vec{\nabla}\varphi_e=0,\\
&\left[\vec{\nabla}\times\vec{A}_e\right]=0,\\
&\frac{1}{c}\frac{\partial\varphi_g}{\partial t}+\left(\vec{\nabla}\cdot\vec{A}_g\right)=0,\\
&\frac{1}{c}\frac{\partial\vec{A}_g}{\partial t}+\vec{\nabla}\varphi_g=0,\\
&\left[\vec{\nabla}\times\vec{A}_g\right]=0.
\end{aligned} \tag{9.12}$$

Thus, one can assume that the generalized equation (9.5) describes the special field of a gravitoelectromagnetic nature. The potentials φ_e and $\vec{A}_e$

describe the electromagnetic component, while the potentials φ_g and $\vec{A}_g$ describe the gravitational component of the neutrino field.

9.2. Second-order relations for neutrino field

Multiplying the expression (9.5) on potential $\tilde{\mathbf{W}}_\nu$ from the left, we obtain the following sedeonic equation:

$$\left(i\mathbf{e}_\mathbf{t}\varphi_\nu + \mathbf{e}_\mathbf{r}\vec{A}_\nu\right)\left(i\mathbf{e}_\mathbf{t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_\mathbf{r}\vec{\nabla}\right)\left(i\mathbf{e}_\mathbf{t}\varphi_\nu + \mathbf{e}_\mathbf{r}\vec{A}_\nu\right) = 0. \tag{9.13}$$

Performing the sedeonic multiplication and separating different terms we get second order expressions for the neutrino field potentials:

$$\frac{1}{2c}\frac{\partial}{\partial t}\left\{\varphi_\nu^2 + \vec{A}_\nu^2\right\} + \left(\vec{\nabla}\cdot\varphi_\nu\vec{A}_\nu\right) = 0, \tag{9.14}$$

$$\left(\vec{A}_\nu\cdot\left[\vec{\nabla}\times\vec{A}_\nu\right]\right) = 0\,, \tag{9.15}$$

$$\frac{1}{c}\left[\vec{A}_\nu\times\frac{\partial\vec{A}_\nu}{\partial t}\right] + \left[\varphi_\nu\vec{\nabla}\times\vec{A}_\nu\right] + \left[\vec{A}_\nu\times\vec{\nabla}\varphi_\nu\right] = 0\,, \tag{9.16}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left\{\varphi_\nu\vec{A}_\nu\right\} + \frac{1}{2}\vec{\nabla}\left\{\varphi_\nu^2 - \vec{A}_\nu^2\right\} + \left(\vec{\nabla}\cdot\vec{A}_\nu\right)\vec{A}_\nu = 0. \tag{9.17}$$

Separating the real and imaginary parts and excluding the cross-terms (taking into account that $\boldsymbol{\varepsilon}_e\boldsymbol{\varepsilon}_g = 0$) we get following four equations:

$$\frac{1}{2c}\frac{\partial}{\partial t}\left\{\varphi_e^2 + \vec{A}_e^2 - \varphi_g^2 - \vec{A}_g^2\right\} + \left(\vec{\nabla}\cdot\varphi_e\vec{A}_e\right) - \left(\vec{\nabla}\cdot\varphi_g\vec{A}_g\right) = 0, \tag{9.18}$$

$$\begin{gathered}\frac{1}{2}\vec{\nabla}\left\{\varphi_e^2 - \vec{A}_e^2 - \varphi_g^2 + \vec{A}_g^2\right\} + \frac{1}{c}\frac{\partial}{\partial t}\left\{\varphi_e\vec{A}_e - \varphi_g\vec{A}_g\right\} \\ + \left(\vec{\nabla}\cdot\vec{A}_e\right)\vec{A}_e - \left(\vec{\nabla}\cdot\vec{A}_g\right)\vec{A}_g = 0,\end{gathered} \tag{9.19}$$

$$\left(\vec{A}_e\cdot\left[\vec{\nabla}\times\vec{A}_e\right]\right) - \left(\vec{A}_g\cdot\left[\vec{\nabla}\times\vec{A}_g\right]\right) = 0, \tag{9.20}$$

$$\frac{1}{c}\left\{\left[\vec{A}_e \times \frac{\partial \vec{A}_e}{\partial t}\right] - \left[\vec{A}_g \times \frac{\partial \vec{A}_g}{\partial t}\right]\right\}$$
$$+\left[\varphi_e \vec{\nabla} \times \vec{A}_e\right] - \left[\varphi_g \vec{\nabla} \times \vec{A}_g\right] + \left[\vec{A}_e \times \vec{\nabla}\varphi_e\right] - \left[\vec{A}_g \times \vec{\nabla}\varphi_g\right] = 0. \quad (9.21)$$

On the other hand, multiplying the expression (9.5) on $\left(-i\mathbf{e_t}\varphi_v + \mathbf{e_r}\vec{A}_v\right)$ from the left, we obtain the following sedeonic equation:

$$\left(-i\mathbf{e_t}\varphi_v + \mathbf{e_r}\vec{A}_v\right)\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla}\right)\left(i\mathbf{e_t}\varphi_v + \mathbf{e_r}\vec{A}_v\right) = 0. \quad (9.22)$$

Performing the sedeonic multiplication and separating different terms we get following expressions

$$\frac{1}{2c}\frac{\partial}{\partial t}\left\{\varphi_v^2 - \vec{A}_v^2\right\} + \varphi_v\left(\vec{\nabla}\cdot\vec{A}_v\right) - \left(\vec{A}_v\cdot\vec{\nabla}\right)\varphi_v = 0, \quad (9.23)$$

$$\left(\vec{A}_v\cdot\left[\vec{\nabla}\times\vec{A}_v\right]\right) = 0, \quad (9.24)$$

$$\frac{1}{c}\left[\vec{A}_v \times \frac{\partial \vec{A}_v}{\partial t}\right] - \left[\vec{\nabla}\times\varphi_v\vec{A}_v\right] = 0, \quad (9.25)$$

$$\frac{1}{2}\vec{\nabla}\left\{\varphi_v^2 + \vec{A}_v^2\right\} + \frac{1}{c}\left\{\varphi_v\frac{\partial \vec{A}_v}{\partial t} - \vec{A}_v\frac{\partial \varphi_v}{\partial t}\right\} - \left(\vec{\nabla}\cdot\vec{A}_v\right)\vec{A}_v = 0. \quad (9.26)$$

Separating the real and imaginary parts and excluding the cross-terms (taking into account that $\boldsymbol{\varepsilon}_e\boldsymbol{\varepsilon}_g = 0$) we get another four equations:

$$\frac{1}{2c}\frac{\partial}{\partial t}\left\{\varphi_e^2 - \vec{A}_e^2 - \varphi_g^2 + \vec{A}_g^2\right\}$$
$$+\varphi_e\left(\vec{\nabla}\cdot\vec{A}_e\right) + \varphi_g\left(\vec{\nabla}\cdot\vec{A}_g\right) - \left(\vec{A}_e\cdot\vec{\nabla}\right)\varphi_e - \left(\vec{A}_g\cdot\vec{\nabla}\right)\varphi_g = 0, \quad (9.27)$$

$$\frac{1}{2}\vec{\nabla}\left\{\varphi_e^2 + \vec{A}_e^2 - \varphi_g^2 - \vec{A}_g^2\right\}$$
$$+\frac{1}{c}\left\{\varphi_e\frac{\partial \vec{A}_e}{\partial t} - \vec{A}_e\frac{\partial \varphi_e}{\partial t} + \vec{A}_g\frac{\partial \varphi_g}{\partial t} - \varphi_g\frac{\partial \vec{A}_g}{\partial t}\right\} \quad (9.28)$$
$$-\left(\vec{\nabla}\cdot\vec{A}_e\right)\vec{A}_e - \left(\vec{\nabla}\cdot\vec{A}_g\right)\vec{A}_g = 0,$$

$$\left(\vec{A}_e\cdot\left[\vec{\nabla}\times\vec{A}_e\right]\right) - \left(\vec{A}_g\cdot\left[\vec{\nabla}\times\vec{A}_g\right]\right) = 0, \quad (9.29)$$

$$\frac{1}{c}\left\{\left[\vec{A}_e \times \frac{\partial \vec{A}_e}{\partial t}\right] - \left[\vec{A}_g \times \frac{\partial \vec{A}_g}{\partial t}\right]\right\} - \left[\vec{\nabla} \times \varphi_e \vec{A}_e\right] + \left[\vec{\nabla} \times \varphi_g \vec{A}_g\right] = 0. \qquad (9.30)$$

The expressions (9.18), (9.19), (9.27) and (9.28) are the analogs of Poynting theorem and Lorentz invariants relations for the neutrino field.

9.3. Plane wave solution for the first-order equation

The first-order wave equation for the neutrino field

$$\left(i\mathbf{e}_\mathbf{t} \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_\mathbf{r}\vec{\nabla}\right)\tilde{\mathbf{W}}_\nu = 0 \qquad (9.31)$$

has the solution in the form of plane wave:

$$\tilde{\mathbf{W}}_\nu = \tilde{\mathbf{U}}_\nu \exp\left\{-i\omega t + i\left(\vec{k} \cdot \vec{r}\right)\right\}. \qquad (9.32)$$

where ω is a frequency, $\vec{k}$ is an absolute wave vector and the wave amplitude $\tilde{\mathbf{U}}_\nu$ does not depend on coordinates and time. In this case the dependence of the frequency on the wave vector has two branches:

$$\omega_\pm = \pm ck , \qquad (9.33)$$

where k is the modulus of wave vector ($k = |\vec{k}|$). The solution of equation (9.31) in the form of a plane wave can be obtained directly from the solution of the first-order equation for a massive field (8.44), equating the mass of the quantum of field to zero. In general, the solution of equation (9.31) can be written as a plane wave of the following form:

$$\tilde{\mathbf{W}}_\nu = \left(\mathbf{e}_1 \frac{\omega_\pm}{c} - i\mathbf{e}_2 \vec{k}\right)\tilde{\mathbf{M}}_\nu \exp\left\{-i\omega_\pm t + i\left(\vec{k} \cdot \vec{r}\right)\right\}, \qquad (9.34)$$

where $\tilde{\mathbf{M}}_\nu$ is arbitrary sedeon with constant components, which do not depend on coordinates and time.

Let us analyze the structure of the plane wave solution (9.34) in detail. Note that the internal structure of this wave is changed under space and time conjugation. Further we suppose that wave vector $\vec{k}$ is directed along z axis. Then the first-order equation (9.31) can be rewritten in the following equivalent form:

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{e}_{\mathbf{tr}}\mathbf{a}_3\frac{\partial}{\partial z}\right)\tilde{\mathbf{W}}'_\nu=0\,, \qquad (9.35)$$

where $\tilde{\mathbf{W}}'_\nu=i\mathbf{e}_\mathbf{t}\tilde{\mathbf{W}}_\nu$. The solution of (9.35) can be presented in form of two waves:

$$\tilde{\mathbf{W}}'_{\nu+}=-\left(1+\mathbf{e}_{\mathbf{tr}}\mathbf{a}_3\right)k\tilde{\mathbf{M}}_\nu\exp\left\{-i\omega_+t+i\left(\vec{k}\cdot\vec{r}\right)\right\}, \qquad (9.36)$$

$$\tilde{\mathbf{W}}'_{\nu-}=\left(1-\mathbf{e}_{\mathbf{tr}}\mathbf{a}_3\right)k\tilde{\mathbf{M}}_\nu\exp\left\{-i\omega_-t+i\left(\vec{k}\cdot\vec{r}\right)\right\}. \qquad (9.37)$$

Note that the wave function $\tilde{\mathbf{W}}'_{\nu+}$ corresponds to the positive branch of dispersion law (9.33) and describes the particle with positive energy, while $\tilde{\mathbf{W}}'_{\nu-}$ corresponds to the negative branch of dispersion law (9.33) and describes the particle with negative energy. Besides, the wave functions (9.36) and (9.37) are the eigenfunctions of spin operator

$$\hat{S}_z=\frac{1}{2}\mathbf{e}_{\mathbf{tr}}\mathbf{a}_3\,. \qquad (9.38)$$

Indeed, it is simple to check that $\tilde{\mathbf{W}}'_\nu$ satisfies the following equation:

$$\hat{S}_z\tilde{\mathbf{W}}'_\nu=S_z\tilde{\mathbf{W}}'_\nu\,, \qquad (9.39)$$

where eigenvalue $S_z=\pm1/2$. Thus, the wave $\tilde{\mathbf{W}}'_{\nu+}$ describes the particle with spirality $S_z=+1/2$, while $\tilde{\mathbf{W}}'_{\nu-}$ describes the particle with spirality $S_z=-1/2$.

9.4. Scalar neutrino source

Let us consider the nonhomogeneous equation of neutrino field

$$\left(i\mathbf{e}_\mathbf{t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_\mathbf{r}\vec{\nabla}\right)\tilde{\mathbf{W}}_\nu=\tilde{\mathbf{I}}_\nu\,, \qquad (9.40)$$

where $\tilde{\mathbf{I}}_\nu$ is phenomenological source. We choose the scalar source in the form

$$\tilde{\mathbf{I}}_\nu=-4\pi\sigma_\nu\,, \qquad (9.41)$$

where σ_ν is the density of neutrino charge and has two components:

$$\sigma_\nu = \sigma_{\nu e} + i\sigma_{\nu g} . \tag{9.42}$$

Choosing the potential $\tilde{\mathbf{W}}_\nu$ in the form (9.2):

$$\tilde{\mathbf{W}}_\nu = i\mathbf{e_t}\varphi_\nu + \mathbf{e_r}\vec{A}_\nu , \tag{9.43}$$

we obtain following equation for the neutrino field:

$$\left(i\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla}\right)\left(i\mathbf{e_t}\varphi_\nu + \mathbf{e_r}\vec{A}_\nu\right) = -4\pi\sigma_\nu . \tag{9.44}$$

It follows that only scalar field strength f_ν is nonzero:

$$f_\nu = 4\pi\sigma_\nu . \tag{9.45}$$

The density of neutrino charge for point source is equal

$$\sigma_\nu = q_\nu \delta(\vec{r}) , \tag{9.46}$$

where q_ν is point neutrino charge:

$$q_\nu = q_{\nu e} + iq_{\nu g} . \tag{9.47}$$

Then the interaction energy of two point neutrino charges can be represented as follows:

$$W_{\nu 1 \nu 2} = \frac{1}{4\pi}\int f_{\nu 1} f_{\nu 2} dV . \tag{9.48}$$

Substituting (9.45) and (9.46), we obtain

$$W_{\nu 1 \nu 2} = 4\pi\left(q_{\nu e1} q_{\nu e2} - q_{\nu g1} q_{\nu g2}\right)\delta(\vec{R}) , \tag{9.49}$$

where $\vec{R}$ is the vector of distance between first and second charges.

9.5. Conclusion

Thus, in this chapter we have developed a description of massless neutrino field based on space-time algebra of sixteen-component sedeons. We have derived the second-order relations for the neutrino potentials, which are analogues to the Pointing theorem and Lorentz invariants relations for gravito-electromagnetic field. The plane wave solution of first-order wave equation for massless field is considered. We also derived the expression for the interaction energy of point neutrino charges.

Chapter 10. Supersymmetric field equations

In classical electrodynamics the electromagnetic field is described by scalar φ and vector $\vec{A}$ potentials [26]. The strengths of electric and magnetic fields are defined as:

$$\begin{aligned} \vec{E} &= -\partial\vec{A} - \vec{\nabla}\varphi, \\ \vec{H} &= \left[\vec{\nabla}\times\vec{A}\right]. \end{aligned} \qquad (10.1)$$

Here $\vec{\nabla}$ is the Hamilton operator (nabla-operator) and we use the following notation for the time differential operator:

$$\partial = \frac{1}{c}\frac{\partial}{\partial t}, \qquad (10.2)$$

where c is the speed of light. The electromagnetic field potentials satisfy the Lorentz gauge condition

$$\partial\varphi + \left(\vec{\nabla}\cdot\vec{A}\right) = 0\,. \qquad (10.3)$$

The equations for electromagnetic field are gauge-invariant. The substitutions

$$\begin{aligned} \varphi &\to \varphi + \partial\alpha, \\ \vec{A} &\to \vec{A} - \vec{\nabla}\alpha, \end{aligned} \qquad (10.4)$$

do not change the electric and magnetic fields. Here $\alpha(\vec{r},t)$ is arbitrary scalar function satisfying homogeneous wave equation (because of the Lorentz gauge (10.3)). The gauge invariance is a cornerstone of modern field theory. However, if the mass of a field quantum is nonzero (massive field), there is a problem with the violation of gauge invariance.

In present part, we use the sedeonic approach for the construction of symmetric equations for massive and massless fields. The gauge invariance of supersymmetric sedeonic field equations is demonstrated.

10.1. Supersymmetric second-order equation for massive field

Let us consider the sedeonic second-order wave equation for massive field [19]:

$$\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\tilde{\mathbf{W}}_\mathbf{m}=\tilde{\mathbf{J}}_\mathbf{m}. \qquad (10.5)$$

where $\tilde{\mathbf{W}}_\mathbf{m}$ is a sedeonic potential, $\tilde{\mathbf{J}}_\mathbf{m}$ is a phenomenological sedeonic source of massive field (index **m**). We use the following operators:

$$\partial=\frac{1}{c}\frac{\partial}{\partial t},$$
$$\vec{\nabla}=\frac{\partial}{\partial x}\mathbf{a_1}+\frac{\partial}{\partial y}\mathbf{a_2}+\frac{\partial}{\partial z}\mathbf{a_3}, \qquad (10.6)$$
$$m=\frac{m_0c}{\hbar}.$$

Let us choose the potential as

$$\tilde{\mathbf{W}}_\mathbf{m}=ia_1\mathbf{e_t}-ia_2\mathbf{e_r}+a_3-ia_4\mathbf{e_{tr}}+\vec{A}_1\mathbf{e_r}+\vec{A}_2\mathbf{e_t}-\vec{A}_3\mathbf{e_{tr}}+i\vec{A}_4, \qquad (10.7)$$

where components a_S and $\vec{A}_\mathrm{S}$ are real functions of coordinates and time. Here and further the index S = 1, 2, 3, 4. Also we take the source in the following form:

$$\tilde{\mathbf{J}}_\mathbf{m}=-i\rho_1\mathbf{e_t}+i\rho_2\mathbf{e_r}-\rho_3+i\rho_4\mathbf{e_{tr}}-\vec{j}_1\mathbf{e_r}-\vec{j}_2\mathbf{e_t}+\vec{j}_3\mathbf{e_{tr}}-\vec{j}_4 i, \qquad (10.8)$$

where $\rho_\mathrm{S}=4\pi\rho'_\mathrm{S}$ (ρ'_k is the volume density of charge) and $\vec{j}_\mathrm{S}=\frac{4\pi}{c}\vec{j}'_\mathrm{S}$ ($\vec{j}'_\mathrm{S}$ is volume density of current). Multiplying the operators in the left part of equation (10.5) we obtain the following wave equations for the components of potentials:

$$\left(\partial^2-\Delta+m^2\right)a_\mathrm{S}=\rho_\mathrm{S},$$
$$\left(\partial^2-\Delta+m^2\right)\vec{A}_\mathrm{S}=\vec{j}_\mathrm{S}. \qquad (10.9)$$

Let us introduce the scalar g_S and vector $\vec{G}_\mathrm{S}$ field strengths according the following definitions:

$$
\begin{aligned}
g_1 &= \partial a_1 + \left(\vec{\nabla} \cdot \vec{A}_1\right) + m a_4, \\
g_2 &= \partial a_2 + \left(\vec{\nabla} \cdot \vec{A}_2\right) - m a_3, \\
g_3 &= \partial a_3 + \left(\vec{\nabla} \cdot \vec{A}_3\right) + m a_2, \\
g_4 &= \partial a_4 + \left(\vec{\nabla} \cdot \vec{A}_4\right) - m a_1, \\
\vec{G}_1 &= -\partial \vec{A}_1 - \vec{\nabla} a_1 + i\left[\vec{\nabla} \times \vec{A}_2\right] + m\vec{A}_4, \\
\vec{G}_2 &= -\partial \vec{A}_2 - \vec{\nabla} a_2 - i\left[\vec{\nabla} \times \vec{A}_1\right] - m\vec{A}_3, \\
\vec{G}_3 &= -\partial \vec{A}_3 - \vec{\nabla} a_3 - i\left[\vec{\nabla} \times \vec{A}_4\right] + m\vec{A}_2, \\
\vec{G}_4 &= -\partial \vec{A}_4 - \vec{\nabla} a_4 + i\left[\vec{\nabla} \times \vec{A}_3\right] - m\vec{A}_1.
\end{aligned} \tag{10.10}
$$

The definitions of field strengths (10.10) have the specific gauge invariance. It is easy to verify that g_S and $\vec{G}_S$ are not changed under the following substitutions for the potentials:

$$
\begin{aligned}
a_1 &\Rightarrow a_1 + \partial \varepsilon_1 - m\varepsilon_4, \\
a_2 &\Rightarrow a_2 + \partial \varepsilon_2 + m\varepsilon_3, \\
a_3 &\Rightarrow a_3 + \partial \varepsilon_3 - m\varepsilon_2, \\
a_4 &\Rightarrow a_4 + \partial \varepsilon_4 + m\varepsilon_1, \\
\vec{A}_1 &\Rightarrow \vec{A}_1 - \vec{\nabla} \varepsilon_1, \\
\vec{A}_2 &\Rightarrow \vec{A}_2 - \vec{\nabla} \varepsilon_2, \\
\vec{A}_3 &\Rightarrow \vec{A}_3 - \vec{\nabla} \varepsilon_3, \\
\vec{A}_4 &\Rightarrow \vec{A}_4 - \vec{\nabla} \varepsilon_4.
\end{aligned} \tag{10.11}
$$

Here ε_1, ε_2, ε_3, ε_4 are arbitrary scalar functions satisfying the homogeneous Klein-Gordon wave equation. Taking into account (10.10) we get that

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)\left(ia_1\mathbf{e_t} - ia_2\mathbf{e_r} + a_3 - ia_4\mathbf{e_{tr}} + \vec{A}_1\mathbf{e_r} + \vec{A}_2\mathbf{e_t} - \vec{A}_3\mathbf{e_{tr}} + i\vec{A}_4\right) \\
&= -g_1 + ig_2\mathbf{e_{tr}} + ig_3\mathbf{e_t} - ig_4\mathbf{e_r} + \vec{G}_1\mathbf{e_{tr}} - i\vec{G}_2 + \vec{G}_3\mathbf{e_r} + \vec{G}_4\mathbf{e_t},
\end{aligned} \tag{10.12}
$$

and the initial wave equation (10.5) is reduced to the following equation:

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\left(-g_1+ig_2\mathbf{e_{tr}}+ig_3\mathbf{e_t}-ig_4\mathbf{e_r}+\vec{G}_1\mathbf{e_{tr}}-i\vec{G}_2+\vec{G}_3\mathbf{e_r}+\vec{G}_4\mathbf{e_t}\right)\\
&=-i\rho_1\mathbf{e_t}+i\rho_2\mathbf{e_r}-\rho_3+i\rho_4\mathbf{e_{tr}}-\vec{j}_1\mathbf{e_r}-\vec{j}_2\mathbf{e_t}+\vec{j}_3\mathbf{e_{tr}}-\vec{j}_4 i.
\end{aligned}
\tag{10.13}
$$

Producing the action of the operator on the left side of equation (10.13) and separating the values with different space-time properties, we obtain a system of equations for the field strengths, similar to the system of Maxwell equations in electrodynamics:

$$
\begin{aligned}
&\partial g_1+\left(\vec{\nabla}\cdot\vec{G}_1\right)-mg_4=\rho_1\,,\\
&\partial g_2+\left(\vec{\nabla}\cdot\vec{G}_2\right)+mg_3=\rho_2,\\
&\partial g_3+\left(\vec{\nabla}\cdot\vec{G}_3\right)-mg_2=\rho_3,\\
&\partial g_4+\left(\vec{\nabla}\cdot\vec{G}_4\right)+mg_1=\rho_4,\\
&\partial\vec{G}_1+\vec{\nabla}g_1+i\left[\vec{\nabla}\times\vec{G}_2\right]+m\vec{G}_4=-\vec{j}_1,\\
&\partial\vec{G}_2+\vec{\nabla}g_2-i\left[\vec{\nabla}\times\vec{G}_1\right]-m\vec{G}_3=-\vec{j}_2,\\
&\partial\vec{G}_3+\vec{\nabla}g_3-i\left[\vec{\nabla}\times\vec{G}_4\right]+m\vec{G}_2=-\vec{j}_3,\\
&\partial\vec{G}_4+\vec{\nabla}g_4+i\left[\vec{\nabla}\times\vec{G}_3\right]-m\vec{G}_1=-\vec{j}_4.
\end{aligned}
\tag{10.14}
$$

The system (10.14) is also invariant with respect to the following substitutions:

$$
\begin{aligned}
&g_1\Rightarrow g_1+\partial\varepsilon_1-m\varepsilon_4,\\
&g_2\Rightarrow g_2-\partial\varepsilon_2-m\varepsilon_3,\\
&g_3\Rightarrow g_3+\partial\varepsilon_3-m\varepsilon_2,\\
&g_4\Rightarrow g_4-\partial\varepsilon_4-m\varepsilon_1,\\
&\vec{G}_1\Rightarrow\vec{G}_1-\vec{\nabla}\varepsilon_1,\\
&\vec{G}_2\Rightarrow\vec{G}_2+\vec{\nabla}\varepsilon_2,\\
&\vec{G}_3\Rightarrow\vec{G}_3-\vec{\nabla}\varepsilon_3,\\
&\vec{G}_4\Rightarrow\vec{G}_4+\vec{\nabla}\varepsilon_4,
\end{aligned}
\tag{10.15}
$$

Multiplying each of the equations (10.14) to the corresponding field strength and adding these equations to each other, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\partial\left(g_1^2+g_2^2+g_3^2+g_4^2+\vec{G}_1^2+\vec{G}_2^2+\vec{G}_3^2+\vec{G}_4^2\right)\\
&+g_1\left(\vec{\nabla}\cdot\vec{G}_1\right)+g_2\left(\vec{\nabla}\cdot\vec{G}_2\right)+g_3\left(\vec{\nabla}\cdot\vec{G}_3\right)+g_4\left(\vec{\nabla}\cdot\vec{G}_4\right)\\
&+\left(\vec{G}_1\cdot\vec{\nabla}g_1\right)+\left(\vec{G}_2\cdot\vec{\nabla}g_2\right)+\left(\vec{G}_3\cdot\vec{\nabla}g_3\right)+\left(\vec{G}_4\cdot\vec{\nabla}g_4\right)\\
&+i\left(\vec{G}_1\cdot\left[\vec{\nabla}\times\vec{G}_2\right]\right)-i\left(\vec{G}_2\cdot\left[\vec{\nabla}\times\vec{G}_1\right]\right)-i\left(\vec{G}_3\cdot\left[\vec{\nabla}\times\vec{G}_4\right]\right)+i\left(\vec{G}_4\cdot\left[\vec{\nabla}\times\vec{G}_3\right]\right)\\
&=g_1\rho_1+g_2\rho_2+g_3\rho_3+g_4\rho_4-\left(\vec{G}_1\cdot\vec{j}_1\right)-\left(\vec{G}_2\cdot\vec{j}_2\right)-\left(\vec{G}_3\cdot\vec{j}_3\right)-\left(\vec{G}_4\cdot\vec{j}_4\right).
\end{aligned}
\tag{10.16}
$$

This expression is the analog of Poynting's theorem for massive field. The term

$$
w=\frac{1}{8\pi}\left(g_1^2+g_2^2+g_3^2+g_4^2+\vec{G}_1^2+\vec{G}_2^2+\vec{G}_3^2+\vec{G}_4^2\right)
\tag{10.17}
$$

plays the role of field energy density, while the term

$$
\vec{p}=\frac{c}{4\pi}\left(g_1\vec{G}_1+g_2\vec{G}_2+g_3\vec{G}_3+g_4\vec{G}_4-i\left[\vec{G}_1\times\vec{G}_2\right]+i\left[\vec{G}_3\times\vec{G}_4\right]\right)
\tag{10.18}
$$

plays the role of energy flux density.

On the other hand, applying the operator $\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)$ to the equation (10.13) we obtain the following wave equation for the field strengths:

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\\
&\times\left(-g_1+ig_2\mathbf{e_{tr}}+ig_3\mathbf{e_t}-ig_4\mathbf{e_r}+\vec{G}_1\mathbf{e_{tr}}-i\vec{G}_2+\vec{G}_3\mathbf{e_r}+\vec{G}_4\mathbf{e_t}\right)\\
&=\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\\
&\times\left(-i\rho_1\mathbf{e_t}+i\rho_2\mathbf{e_r}-\rho_3+i\rho_4\mathbf{e_{tr}}-\vec{j}_1\mathbf{e_r}-\vec{j}_2\mathbf{e_t}+\vec{j}_3\mathbf{e_{tr}}-\vec{j}_4 i\right).
\end{aligned}
\tag{10.19}
$$

Separating the terms with different space-time properties we get the following wave equation for the field strength components g_S and $\vec{G}_S$:

$$
\begin{aligned}
&\left(\partial^2-\Delta+m^2\right)g_1=-\,\partial\rho_1-\left(\vec{\nabla}\cdot\vec{j}_1\right)-m\rho_4,\\
&\left(\partial^2-\Delta+m^2\right)g_2=-\partial\rho_2-\left(\vec{\nabla}\cdot\vec{j}_2\right)+m\rho_3,\\
&\left(\partial^2-\Delta+m^2\right)g_3=-\partial\rho_3-\left(\vec{\nabla}\cdot\vec{j}_3\right)-m\rho_2,\\
&\left(\partial^2-\Delta+m^2\right)g_4=-\partial\rho_4-\left(\vec{\nabla}\cdot\vec{j}_4\right)+m\rho_1,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_1=\vec{\nabla}\rho_1+\partial\vec{j}_1-i\left[\vec{\nabla}\times\vec{j}_2\right]-m\vec{j}_4,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_2=\vec{\nabla}\rho_2+\partial\vec{j}_2+i\left[\vec{\nabla}\times\vec{j}_1\right]+m\vec{j}_3,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_3=\vec{\nabla}\rho_3+\partial\vec{j}_3+i\left[\vec{\nabla}\times\vec{j}_4\right]-m\vec{j}_2,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_4=\vec{\nabla}\rho_4+\partial\vec{j}_4-i\left[\vec{\nabla}\times\vec{j}_3\right]+m\vec{j}_1.
\end{aligned}
\tag{10.20}
$$

It can be seen that equations (10.20) are invariant with respect to the following substitutions:

$$
\begin{aligned}
&\rho_1\Rightarrow\rho_1+\partial\varepsilon_1-m\varepsilon_4,\\
&\rho_2\Rightarrow\rho_2+\partial\varepsilon_2+m\varepsilon_3,\\
&\rho_3\Rightarrow\rho_3+\partial\varepsilon_3-m\varepsilon_2,\\
&\rho_4\Rightarrow\rho_4+\partial\varepsilon_4+m\varepsilon_1,\\
&\vec{j}_1\Rightarrow\vec{j}_1-\vec{\nabla}\varepsilon_1,\\
&\vec{j}_2\Rightarrow\vec{j}_2-\vec{\nabla}\varepsilon_2,\\
&\vec{j}_3\Rightarrow\vec{j}_3-\vec{\nabla}\varepsilon_3,\\
&\vec{j}_4\Rightarrow\vec{j}_4-\vec{\nabla}\varepsilon_4.
\end{aligned}
\tag{10.21}
$$

As an example, let us consider the fields produced by a one type of sources ρ_1 and $\vec{j}_1$. In this case the massive field is described by a_1 and $\vec{A}_1$ potentials:

$$
\tilde{\mathbf{W}}_{\mathbf{m}}=ia_1\mathbf{e_t}+\vec{A}_1\mathbf{e_r}\,.
\tag{10.22}
$$

Then we have only the following nonzero field's strengths:

$$
\begin{aligned}
g_1 &= \partial a_1 + \left(\vec{\nabla} \cdot \vec{A}_1\right), \\
g_4 &= -m a_1, \\
\vec{G}_1 &= -\partial \vec{A}_1 - \vec{\nabla} a_1, \\
\vec{G}_2 &= -i\left[\vec{\nabla} \times \vec{A}_1\right], \\
\vec{G}_4 &= -m \vec{A}_1,
\end{aligned}
\tag{10.23}
$$

and the wave equation (10.13) takes the following form:

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}} m\right)\left(-g_1 - i g_4 \mathbf{e_r} + \vec{G}_1 \mathbf{e_{tr}} - i\vec{G}_2 + \vec{G}_4 \mathbf{e_t}\right) \\
&= -i\rho_1 \mathbf{e_t} - \vec{j}_1 \mathbf{e_r}.
\end{aligned}
\tag{10.24}
$$

Then the system (10.14) can be rewritten as

$$
\begin{aligned}
&\partial g_1 + \left(\vec{\nabla} \cdot \vec{G}_1\right) - m g_4 = \rho_1 , \\
&\left(\vec{\nabla} \cdot \vec{G}_2\right) = 0, \\
&\partial g_4 + \left(\vec{\nabla} \cdot \vec{G}_4\right) + m g_1 = 0, \\
&\partial \vec{G}_1 + \vec{\nabla} g_1 + i\left[\vec{\nabla} \times \vec{G}_2\right] + m\vec{G}_4 = -\vec{j}_1, \\
&\partial \vec{G}_2 - i\left[\vec{\nabla} \times \vec{G}_1\right] = 0, \\
&-i\left[\vec{\nabla} \times \vec{G}_4\right] + m\vec{G}_2 = 0, \\
&\partial \vec{G}_4 + \vec{\nabla} g_4 - m\vec{G}_1 = 0.
\end{aligned}
\tag{10.25}
$$

The system (10.25) is the analog of Proca-Maxwell equations. In addition, we have the following wave equations for the field strengths:

$$
\begin{aligned}
\left(\partial^2 - \Delta + m^2\right) g_1 &= \partial \rho_1 + \left(\vec{\nabla} \cdot \vec{j}_1\right), \\
\left(\partial^2 - \Delta + m^2\right) g_4 &= -m\rho_1, \\
\left(\partial^2 - \Delta + m^2\right) \vec{G}_1 &= -\vec{\nabla}\rho_1 - \partial \vec{j}_1, \\
\left(\partial^2 - \Delta + m^2\right) \vec{G}_2 &= -i\left[\vec{\nabla} \times \vec{j}_1\right], \\
\left(\partial^2 - \Delta + m^2\right) \vec{G}_4 &= -m\vec{j}_1.
\end{aligned}
\tag{10.26}
$$

Assuming the charge conservation

$$\partial\rho_1 + \left(\vec{\nabla}\cdot\vec{j}_1\right) = 0\,, \qquad (10.27)$$

we can choose the scalar field strength g_1 equal to zero. This is equivalent to the following gauge condition:

$$\partial a_1 + \left(\vec{\nabla}\cdot\vec{A}_1\right) = 0\,, \qquad (10.28)$$

similar to the Lorentz gauge in electrodynamics.

10.2. Second-order equation for massless field

In the case of massless field the equation (10.5) takers the following form:

$$\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\tilde{\mathbf{W}}_\mathbf{0} = \tilde{\mathbf{J}}_\mathbf{0}\,, \qquad (10.29)$$

where we choose the potential $\tilde{\mathbf{W}}_\mathbf{0}$ and source $\tilde{\mathbf{J}}_\mathbf{0}$ of massless field (index **0**) in the form of (10.7) and (10.8) as before

$$\tilde{\mathbf{W}}_\mathbf{0} = ib_1\mathbf{e_t} - ib_2\mathbf{e_r} + b_3 - ib_4\mathbf{e_{tr}} + \vec{B}_1\mathbf{e_r} + \vec{B}_2\mathbf{e_t} - \vec{B}_3\mathbf{e_{tr}} + i\vec{B}_4\,, \qquad (10.30)$$

$$\tilde{\mathbf{J}}_\mathbf{0} = -i\beta_1\mathbf{e_t} + i\beta_2\mathbf{e_r} - \beta_3 + i\beta_4\mathbf{e_{tr}} - \vec{l}_1\mathbf{e_r} - \vec{l}_2\mathbf{e_t} + \vec{l}_3\mathbf{e_{tr}} - \vec{l}_4 i\,, \qquad (10.31)$$

where $\beta_S = 4\pi\beta'_S$ (β'_S is the volume density of charge) and $\vec{l}_S = \dfrac{4\pi}{c}\vec{l}'_S$ ($\vec{l}'_S$ is volume density of current). We introduce the scalar and vector field strengths according following definitions:

$$
\begin{aligned}
h_1 &= \partial b_1 + \left(\vec{\nabla}\cdot\vec{B}_1\right),\\
h_2 &= \partial b_2 + \left(\vec{\nabla}\cdot\vec{B}_2\right),\\
h_3 &= \partial b_3 + \left(\vec{\nabla}\cdot\vec{B}_3\right),\\
h_4 &= \partial b_4 + \left(\vec{\nabla}\cdot\vec{B}_4\right)_1,\\
\vec{H}_1 &= -\partial\vec{B}_1 - \vec{\nabla}b_1 + i\left[\vec{\nabla}\times\vec{B}_2\right],\\
\vec{H}_2 &= -\partial\vec{B}_2 - \vec{\nabla}b_2 - i\left[\vec{\nabla}\times\vec{B}_1\right],\\
\vec{H}_3 &= -\partial\vec{B}_3 - \vec{\nabla}b_3 - i\left[\vec{\nabla}\times\vec{B}_4\right],\\
\vec{H}_4 &= -\partial\vec{B}_4 - \vec{\nabla}b_4 + i\left[\vec{\nabla}\times\vec{B}_3\right].
\end{aligned}
\tag{10.32}
$$

Note that the definitions (10.32) are invariant with respect to the following substitutions:

$$
\begin{aligned}
b_1 &\Rightarrow b_1 + \partial\varepsilon_1,\\
b_2 &\Rightarrow b_2 + \partial\varepsilon_2,\\
b_3 &\Rightarrow b_3 + \partial\varepsilon_3,\\
b_4 &\Rightarrow b_4 + \partial\varepsilon_4,\\
\vec{B}_1 &\Rightarrow \vec{B}_1 - \vec{\nabla}\varepsilon_1,\\
\vec{B}_2 &\Rightarrow \vec{B}_2 - \vec{\nabla}\varepsilon_2,\\
\vec{B}_3 &\Rightarrow \vec{B}_3 - \vec{\nabla}\varepsilon_3,\\
\vec{B}_4 &\Rightarrow \vec{B}_4 - \vec{\nabla}\varepsilon_4.
\end{aligned}
\tag{10.33}
$$

Taking into account (10.32) we get

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\left(ib_1\mathbf{e_t} - ib_2\mathbf{e_r} + b_3 - ib_4\mathbf{e_{tr}} + \vec{B}_1\mathbf{e_r} + \vec{B}_2\mathbf{e_t} - \vec{B}_3\mathbf{e_{tr}} + i\vec{B}_4\right)\\
&= -h_1 + ih_2\mathbf{e_{tr}} + ih_3\mathbf{e_t} - ih_4\mathbf{e_r} + \vec{H}_1\mathbf{e_{tr}} - i\vec{H}_2 + \vec{H}_3\mathbf{e_r} + \vec{H}_4\mathbf{e_t},
\end{aligned}
\tag{10.34}
$$

and wave equation (10.27) can be rewritten as

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\left(-h_1 + ih_2\mathbf{e_{tr}} + ih_3\mathbf{e_t} - ih_4\mathbf{e_r} + \vec{H}_1\mathbf{e_{tr}} - i\vec{H}_2 + \vec{H}_3\mathbf{e_r} + \vec{H}_4\mathbf{e_t}\right)\\
&= -i\beta_1\mathbf{e_t} + i\beta_2\mathbf{e_r} - \beta_3 + i\beta_4\mathbf{e_{tr}} - \vec{l}_1\mathbf{e_r} - \vec{l}_2\mathbf{e_t} + \vec{l}_3\mathbf{e_{tr}} - \vec{l}_4 i.
\end{aligned}
\tag{10.35}
$$

Producing the action of the operator on the left side of equation (10.35) and separating the terms with different space-time properties, we obtain two independent systems of the equations for the field strengths, similar to the system of Maxwell equations in electrodynamics. The first system is

$$\begin{aligned} &\partial h_1 + \left(\vec{\nabla}\cdot\vec{H}_1\right) = \beta_1 , \\ &\partial h_2 + \left(\vec{\nabla}\cdot\vec{H}_2\right) = \beta_2 , \\ &\partial\vec{H}_1 + \vec{\nabla}h_1 + i\left[\vec{\nabla}\times\vec{H}_2\right] = -\vec{l}_1 , \\ &\partial\vec{H}_2 + \vec{\nabla}h_2 - i\left[\vec{\nabla}\times\vec{H}_1\right] = -\vec{l}_2 . \end{aligned} \tag{10.36}$$

This system is invariant with respect to the following substitutions:

$$\begin{aligned} &h_1 \Rightarrow h_1 + \partial\varepsilon_1 , \\ &h_2 \Rightarrow h_2 - \partial\varepsilon_2 , \\ &\vec{H}_1 \Rightarrow \vec{H}_1 - \vec{\nabla}\varepsilon_1 , \\ &\vec{H}_2 \Rightarrow \vec{H}_2 + \vec{\nabla}\varepsilon_2 . \end{aligned} \tag{10.37}$$

Multiplying each of the equations (10.36) to the corresponding field strength and adding these equations to each other, we obtain:

$$\begin{aligned} &\frac{1}{2}\partial\left(h_1^2 + h_2^2 + \vec{H}_1^2 + \vec{H}_2^2\right) \\ &+h_1\left(\vec{\nabla}\cdot\vec{H}_1\right) + h_2\left(\vec{\nabla}\cdot\vec{H}_2\right) \\ &+\left(\vec{H}_1\cdot\vec{\nabla}h_1\right) + \left(\vec{H}_2\cdot\vec{\nabla}h_2\right) \\ &+i\left(\vec{H}_1\cdot\left[\vec{\nabla}\times\vec{H}_2\right]\right) - i\left(\vec{H}_2\cdot\left[\vec{\nabla}\times\vec{H}_1\right]\right) \\ &= h_1\beta_1 + h_2\beta_2 - \left(\vec{H}_1\cdot\vec{l}_1\right) - \left(\vec{H}_2\cdot\vec{l}_2\right). \end{aligned} \tag{10.38}$$

This expression is the analog of Poynting's theorem for first type of massless field. The term

$$w = \frac{1}{8\pi}\left(h_1^2 + h_2^2 + \vec{H}_1^2 + \vec{H}_2^2\right) \tag{10.39}$$

plays the role of field energy density, while the term

$$\vec{p} = \frac{c}{4\pi}\left(h_1\vec{H}_1 + h_2\vec{H}_2 - i\left[\vec{H}_1 \times \vec{H}_2\right]\right) \tag{10.40}$$

plays the role of energy flux density.

The second system is

$$\begin{aligned}
&\partial h_3 + \left(\vec{\nabla}\cdot\vec{H}_3\right) = \beta_3, \\
&\partial h_4 + \left(\vec{\nabla}\cdot\vec{H}_4\right) = \beta_4, \\
&\partial\vec{H}_3 + \vec{\nabla}h_3 - i\left[\vec{\nabla}\times\vec{H}_4\right] = -\vec{l}_3, \\
&\partial\vec{H}_4 + \vec{\nabla}h_4 + i\left[\vec{\nabla}\times\vec{H}_3\right] = -\vec{l}_4.
\end{aligned} \tag{10.41}$$

This system is invariant with respect to the following substitutions:

$$\begin{aligned}
&h_3 \Rightarrow h_3 + \partial\varepsilon_3, \\
&h_4 \Rightarrow h_4 - \partial\varepsilon_4, \\
&\vec{H}_3 \Rightarrow \vec{H}_3 - \vec{\nabla}\varepsilon_3, \\
&\vec{H}_4 \Rightarrow \vec{H}_4 + \vec{\nabla}\varepsilon_4.
\end{aligned} \tag{10.42}$$

Multiplying each of the equations (10.41) to the corresponding field strength and adding these equations to each other, we obtain:

$$\begin{aligned}
&\frac{1}{2}\partial\left(h_3^2 + h_4^2 + \vec{H}_3^2 + \vec{H}_4^2\right) \\
&+h_3\left(\vec{\nabla}\cdot\vec{H}_3\right) + h_4\left(\vec{\nabla}\cdot\vec{H}_4\right) \\
&+\left(\vec{H}_3\cdot\vec{\nabla}h_3\right) + \left(\vec{H}_4\cdot\vec{\nabla}h_4\right) \\
&+i\left(\vec{H}_3\cdot\left[\vec{\nabla}\times\vec{H}_4\right]\right) - i\left(\vec{H}_4\cdot\left[\vec{\nabla}\times\vec{H}_3\right]\right) \\
&= h_3\beta_3 + h_4\beta_4 - \left(\vec{H}_3\cdot\vec{l}_3\right) - \left(\vec{H}_4\cdot\vec{l}_4\right).
\end{aligned} \tag{10.43}$$

This expression is the analog of Poynting’s theorem for second type of massless field. The term

$$w = \frac{1}{8\pi}\left(h_3^2 + h_4^2 + \vec{H}_3^2 + \vec{H}_4^2\right) \tag{10.44}$$

plays the role of field energy density, while the term

$$\vec{p} = \frac{c}{4\pi}\left(h_3\vec{H}_3 + h_4\vec{H}_4 - i\left[\vec{H}_3 \times \vec{H}_4\right]\right) \tag{10.45}$$

plays the role of energy flux density.

Accordingly, the wave equations for the massless field strengths are also divided into two independent systems. The first system combines the potentials and sources, which are transformed in accordance with Lorentz transformations of type I (see (2.10))

$$\begin{aligned}
\left(\partial^2 - \Delta\right)h_1 &= -\partial\beta_1 - \left(\vec{\nabla}\cdot\vec{l}_1\right),\\
\left(\partial^2 - \Delta\right)h_2 &= -\partial\beta_2 - \left(\vec{\nabla}\cdot\vec{l}_2\right),\\
\left(\partial^2 - \Delta\right)\vec{H}_1 &= \vec{\nabla}\beta_1 + \partial\vec{l}_1 - i\left[\vec{\nabla}\times\vec{l}_2\right],\\
\left(\partial^2 - \Delta\right)\vec{H}_2 &= \vec{\nabla}\beta_2 + \partial\vec{l}_2 + i\left[\vec{\nabla}\times\vec{l}_1\right].
\end{aligned} \tag{10.46}$$

The second system combines the fields and sources, which are transformed in accordance with Lorentz transformations of type II (see (2.10))

$$\begin{aligned}
\left(\partial^2 - \Delta\right)h_3 &= -\partial\beta_3 - \left(\vec{\nabla}\cdot\vec{l}_3\right),\\
\left(\partial^2 - \Delta\right)h_4 &= -\partial\beta_4 - \left(\vec{\nabla}\cdot\vec{l}_4\right),\\
\left(\partial^2 - \Delta\right)\vec{H}_3 &= \vec{\nabla}\beta_3 + \partial\vec{l}_3 + i\left[\vec{\nabla}\times\vec{l}_4\right],\\
\left(\partial^2 - \Delta\right)\vec{H}_4 &= \vec{\nabla}\beta_4 + \partial\vec{l}_4 - i\left[\vec{\nabla}\times\vec{l}_3\right].
\end{aligned} \tag{10.47}$$

The equations (10.46) and (10.47) are invariant with respect to the substitutions

$$\begin{aligned}
&\beta_1 \Rightarrow \beta_1 + \partial\varepsilon_1,\\
&\beta_2 \Rightarrow \beta_2 + \partial\varepsilon_2,\\
&\beta_3 \Rightarrow \beta_3 + \partial\varepsilon_3,\\
&\beta_4 \Rightarrow \beta_4 + \partial\varepsilon_4,\\
&\vec{l}_1 \Rightarrow \vec{l}_1 - \vec{\nabla}\varepsilon_1,\\
&\vec{l}_2 \Rightarrow \vec{l}_2 - \vec{\nabla}\varepsilon_2,\\
&\vec{l}_3 \Rightarrow \vec{l}_3 - \vec{\nabla}\varepsilon_3,\\
&\vec{l}_4 \Rightarrow \vec{l}_4 - \vec{\nabla}\varepsilon_4.
\end{aligned} \qquad (10.48)$$

The system of equations (10.36) corresponds to the usual system of Maxwell equations. Let us show it. If we assume the charge conservation

$$\begin{aligned}
&\partial\beta_1 + \left(\vec{\nabla}\cdot\vec{l}_1\right) = 0,\\
&\partial\beta_2 + \left(\vec{\nabla}\cdot\vec{l}_2\right) = 0,
\end{aligned} \qquad (10.49)$$

then as it follows from (10.46) we can choose the scalar fields h_1 and h_2 equal to zero and obtain the following system:

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{H}_1\right) = \beta_1,\\
&\left(\vec{\nabla}\cdot\vec{H}_2\right) = \beta_2,\\
&\partial\vec{H}_1 + i\left[\vec{\nabla}\times\vec{H}_2\right] = -\vec{l}_1,\\
&\partial\vec{H}_2 - i\left[\vec{\nabla}\times\vec{H}_1\right] = -\vec{l}_2.
\end{aligned} \qquad (10.50)$$

Here $\vec{H}_1$ is the electric field strength; $\vec{H}_2$ is the magnetic field strength; β_1 is the volume density of electrical charge; β_2 is the volume density of magnetic charge; $\vec{l}_1$ is the volume density of electrical current; $\vec{l}_2$ is the volume density of magnetic current. Taking into account the experimental fact that in our part of the universe there are no magnetic charges and currents, we obtain the system of equations

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{H}_1\right) = \beta_1,\\
&\left(\vec{\nabla}\cdot\vec{H}_2\right) = 0,\\
&\partial\vec{H}_1 + i\left[\vec{\nabla}\times\vec{H}_2\right] = -\vec{l}_1,\\
&\partial\vec{H}_2 - i\left[\vec{\nabla}\times\vec{H}_1\right] = 0,
\end{aligned} \qquad (10.51)$$

which coincides with the conventional system of Maxwell's equations.

10.3. First-order equation for massive field

Let us consider a massive field, which is described by the sedeonic first-order equation

$$\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)\tilde{\mathbf{W}}_{\mathbf{m}} = \tilde{\mathbf{I}}_{\mathbf{m}} .. \tag{10.52}$$

Here $\tilde{\mathbf{I}}_{\mathbf{m}}$ is the phenomenological field source, which can be chosen in the following sedeonic form:

$$\tilde{\mathbf{I}}_{\mathbf{m}} = -d_1 + id_2\mathbf{e_{tr}} + id_3\mathbf{e_t} - id_4\mathbf{e_r} + \vec{f}_1\mathbf{e_{tr}} - i\vec{f}_2 + \vec{f}_3\mathbf{e_r} + \vec{f}_4\mathbf{e_t} \tag{10.53}$$

where $d_k = 4\pi d'_k$ (d'_k are the volume density of charges) and $\vec{f}_k = \dfrac{4\pi}{c}\vec{f}'_k$ ($\vec{f}'_k$ are the corresponding volume density of currents). Choosing the potential $\tilde{\mathbf{W}}_{\mathbf{m}}$ in the form of (10.7) we can rewrite the equation (10.52) in the following expanded form

$$\begin{gathered}\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)\left(ia_1\mathbf{e_t} - ia_2\mathbf{e_r} + a_3 - ia_4\mathbf{e_{tr}} + \vec{A}_1\mathbf{e_r} + \vec{A}_2\mathbf{e_t} - \vec{A}_3\mathbf{e_{tr}} + i\vec{A}_4\right) \\ = -d_1 + id_2\mathbf{e_{tr}} + id_3\mathbf{e_t} - id_4\mathbf{e_r} + \vec{f}_1\mathbf{e_{tr}} - i\vec{f}_2 + \vec{f}_3\mathbf{e_r} + \vec{f}_4\mathbf{e_t} .\end{gathered} \tag{10.54}$$

This sedeonic equation is equivalent to the following system:

$$
\begin{aligned}
&\partial a_1 + \left(\vec{\nabla}\cdot\vec{A}_1\right) + m a_4 = d_1,\\
&\partial a_2 + \left(\vec{\nabla}\cdot\vec{A}_2\right) - m a_3 = d_2,\\
&\partial a_3 + \left(\vec{\nabla}\cdot\vec{A}_3\right) + m a_2 = d_3,\\
&\partial a_4 + \left(\vec{\nabla}\cdot\vec{A}_4\right) - m a_1 = d_4,\\
&-\partial\vec{A}_1 - \vec{\nabla}a_1 + i\left[\vec{\nabla}\times\vec{A}_2\right] + m\vec{A}_4 = \vec{f}_1,\\
&-\partial\vec{A}_2 - \vec{\nabla}a_2 - i\left[\vec{\nabla}\times\vec{A}_1\right] - m\vec{A}_3 = \vec{f}_2,\\
&-\partial\vec{A}_3 - \vec{\nabla}a_3 - i\left[\vec{\nabla}\times\vec{A}_4\right] + m\vec{A}_2 = \vec{f}_3,\\
&-\partial\vec{A}_4 - \vec{\nabla}a_4 + i\left[\vec{\nabla}\times\vec{A}_3\right] - m\vec{A}_1 = \vec{f}_4.
\end{aligned}
\tag{10.55}
$$

On the other hand, introducing the massive field strengths according the definitions (10.10) we get

$$
\begin{aligned}
&-g_1 + ig_2\mathbf{e_{tr}} + ig_3\mathbf{e_t} - ig_4\mathbf{e_r} + \vec{G}_1\mathbf{e_{tr}} - i\vec{G}_2 + \vec{G}_3\mathbf{e_r} + \vec{G}_4\mathbf{e_t}\\
&= -d_1 + id_2\mathbf{e_{tr}} + id_3\mathbf{e_t} - id_4\mathbf{e_r} + \vec{f}_1\mathbf{e_{tr}} - i\vec{f}_2 + \vec{f}_3\mathbf{e_r} + \vec{f}_4\mathbf{e_t}.
\end{aligned}
\tag{10.56}
$$

It means that in fact the field strengths are non-zero only in the regions of the field sources.

Applying the operator $\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)$ to the equation (10.54) we obtain the following second-order wave equation:

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)\\
&\times\left(ia_1\mathbf{e_t} - ia_2\mathbf{e_r} + a_3 - ia_4\mathbf{e_{tr}} + \vec{A}_1\mathbf{e_r} + \vec{A}_2\mathbf{e_t} - \vec{A}_3\mathbf{e_{tr}} + i\vec{A}_4\right)\\
&= \left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)\\
&\times\left(-d_1 + id_2\mathbf{e_{tr}} + id_3\mathbf{e_t} - id_4\mathbf{e_r} + \vec{f}_1\mathbf{e_{tr}} - i\vec{f}_2 + \vec{f}_3\mathbf{e_r} + \vec{f}_4\mathbf{e_t}\right),
\end{aligned}
\tag{10.57}
$$

which is equivalent to the following system:

$$
\begin{aligned}
&\left(\partial^2-\Delta+m^2\right)a_1=\partial d_1+\left(\vec{\nabla}\cdot\vec{f}_1\right)-md_4,\\
&\left(\partial^2-\Delta+m^2\right)a_2=\partial d_2+\left(\vec{\nabla}\cdot\vec{f}_2\right)+md_3,\\
&\left(\partial^2-\Delta+m^2\right)a_3=\partial d_3+\left(\vec{\nabla}\cdot\vec{f}_3\right)-md_2,\\
&\left(\partial^2-\Delta+m^2\right)a_4=\partial d_4+\left(\vec{\nabla}\cdot\vec{f}_4\right)+md_1,\\
&\left(\partial^2-\Delta+m^2\right)\vec{A}_1=-\partial\vec{f}_1-\vec{\nabla}d_1-i\left[\vec{\nabla}\times\vec{f}_2\right]-m\vec{f}_4,\\
&\left(\partial^2-\Delta+m^2\right)\vec{A}_2=-\partial\vec{f}_2-\vec{\nabla}d_2+i\left[\vec{\nabla}\times\vec{f}_1\right]+m\vec{f}_3,\\
&\left(\partial^2-\Delta+m^2\right)\vec{A}_3=-\partial\vec{f}_3-\vec{\nabla}d_3+i\left[\vec{\nabla}\times\vec{f}_4\right]-m\vec{f}_2,\\
&\left(\partial^2-\Delta+m^2\right)\vec{A}_4=-\partial\vec{f}_4-\vec{\nabla}d_4-i\left[\vec{\nabla}\times\vec{f}_3\right]+m\vec{f}_1.
\end{aligned}
\tag{10.58}
$$

It can be seen that equations (10.58) are invariant with respect to the following substitutions for the sources:

$$
\begin{aligned}
&d_1\Rightarrow d_1+\partial\varepsilon_1+m\varepsilon_4,\\
&d_2\Rightarrow d_2+\partial\varepsilon_2-m\varepsilon_3,\\
&d_3\Rightarrow d_3+\partial\varepsilon_3+m\varepsilon_2,\\
&d_4\Rightarrow d_4+\partial\varepsilon_4-m\varepsilon_1,\\
&\vec{f}_1\Rightarrow\vec{f}_1-\vec{\nabla}\varepsilon_1,\\
&\vec{f}_2\Rightarrow\vec{j}_2-\vec{\nabla}\varepsilon_2,\\
&\vec{f}_3\Rightarrow\vec{f}_3-\vec{\nabla}\varepsilon_3,\\
&\vec{f}_4\Rightarrow\vec{f}_4-\vec{\nabla}\varepsilon_4.
\end{aligned}
\tag{10.59}
$$

As an example, let us consider the fields produced by a one type of sources d_4 and $\vec{f}_4$:

$$
\tilde{\mathbf{I}}_{\mathbf{m}}=-id_4\mathbf{e}_\mathbf{r}+\vec{f}_4\mathbf{e}_\mathbf{t}\,, \tag{10.60}
$$

In this case the equation (10.56) is rewritten as

$$
-ig_4\mathbf{e}_\mathbf{r}+\vec{G}_4\mathbf{e}_\mathbf{t}=-id_4\mathbf{e}_\mathbf{r}+\vec{f}_4\mathbf{e}_\mathbf{t}\,. \tag{10.61}
$$

Applying the operator $\left(i\mathbf{e}_1\partial-\mathbf{e}_2\vec{\nabla}-i\mathbf{e}_3m\right)$ to the equation (10.6) and separating the values with different space-time properties we obtain the following equations for the field strengths:

$$g_4 = d_4,$$
$$\vec{G}_4 = \vec{f}_4,$$
$$\partial g_4 + \left(\vec{\nabla} \cdot \vec{G}_4\right) = \partial d_4 + \frac{1}{c}\left(\vec{\nabla} \cdot \vec{f}_4\right), \tag{10.62}$$
$$\left[\vec{\nabla} \times \vec{G}_4\right] = \left[\vec{\nabla} \times \vec{f}_4\right],$$
$$\partial \vec{G}_4 + \vec{\nabla} g_4 = \partial \vec{f}_4 + \vec{\nabla} d_4.$$

Assuming the charge conservation

$$\partial d_4 + \left(\vec{\nabla} \cdot \vec{f}_4\right) = 0, \qquad (10.63$$

we have the following gauge condition:

$$\partial g_4 + \left(\vec{\nabla} \cdot \vec{G}_4\right) = 0, \qquad (10.64$$

which is similar to conventional Lorentz gauge, but for field strengths here.

10.4. First-order equation for massless field

In massless case the first-order wave equation can be presented as

$$\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\tilde{\mathbf{W}}_{\mathbf{0}} = \tilde{\mathbf{I}}_{\mathbf{0}}, \tag{10.67}$$

where the potential $\tilde{\mathbf{W}}_{\mathbf{0}}$ and phenomenological source $\tilde{\mathbf{I}}_{\mathbf{0}}$ have the following form:

$$\tilde{\mathbf{W}}_{\mathbf{0}} = ib_1\mathbf{e}_{\mathbf{t}} - ib_2\mathbf{e}_{\mathbf{r}} + b_3 - ib_4\mathbf{e}_{\mathbf{tr}} + \vec{B}_1\mathbf{e}_{\mathbf{r}} + \vec{B}_2\mathbf{e}_{\mathbf{t}} - \vec{B}_3\mathbf{e}_{\mathbf{tr}} + i\vec{B}_4, \tag{10.68}$$

$$\tilde{\mathbf{I}}_{\mathbf{0}} = -\upsilon_1 + i\upsilon_2\mathbf{e}_{\mathbf{tr}} + i\upsilon_3\mathbf{e}_{\mathbf{t}} - i\upsilon_4\mathbf{e}_{\mathbf{r}} + \vec{\gamma}_1\mathbf{e}_{\mathbf{tr}} - i\vec{\gamma}_2 + \vec{\gamma}_3\mathbf{e}_{\mathbf{r}} + \vec{\gamma}_4\mathbf{e}_{\mathbf{t}}. \tag{10.69}$$

Here $\upsilon_{\mathrm{s}} = 4\pi\upsilon'_{\mathrm{s}}$ (υ'_{s} is the volume density of charge) and $\vec{\gamma}_{\mathrm{s}} = \frac{4\pi}{c}\vec{\gamma}'_{\mathrm{s}}$ ($\vec{\gamma}'_{\mathrm{s}}$ is volume density of current). The equation (10.67) is equivalent to the following system:

$$
\begin{aligned}
&\partial b_1 + \left(\vec{\nabla} \cdot \vec{B}_1\right) = \upsilon_1, \\
&\partial b_2 + \left(\vec{\nabla} \cdot \vec{B}_2\right) = \upsilon_2, \\
&\partial b_3 + \left(\vec{\nabla} \cdot \vec{B}_3\right) = \upsilon_3, \\
&\partial b_4 + \left(\vec{\nabla} \cdot \vec{B}_4\right) = \upsilon_4, \\
&-\partial \vec{B}_1 - \vec{\nabla} b_1 + i\left[\vec{\nabla} \times \vec{B}_2\right] = \vec{\gamma}_1, \\
&-\partial \vec{B}_2 - \vec{\nabla} b_2 - i\left[\vec{\nabla} \times \vec{B}_1\right] = \vec{\gamma}_2, \\
&-\partial \vec{B}_3 - \vec{\nabla} b_3 - i\left[\vec{\nabla} \times \vec{B}_4\right] = \vec{\gamma}_3, \\
&-\partial \vec{B}_4 - \vec{\nabla} b_4 + i\left[\vec{\nabla} \times \vec{B}_3\right] = \vec{\gamma}_4.
\end{aligned}
\tag{10.70}
$$

The equations (10.70) are invariant with respect to the substitutions (10.33).

10.5. Generalization of gradient invariance

The gradient gauge invariance of the sedeonic equations describing the massive fields is a property of the operator $\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}}m\right)$ and can be generalized to a wider class of scalar-vector substitutions. Indeed, let us denote

$$\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}}m\right) \equiv \hat{\vec{\nabla}}, \tag{10.71}$$

then the wave equation (10.71) takes the following form:

$$\hat{\vec{\nabla}}\hat{\vec{\nabla}}\tilde{\mathbf{W}}_{\mathbf{m}} = \tilde{\mathbf{J}}_{\mathbf{m}}. \tag{10.72}$$

This equation is not changed under the following replacement of potential:

$$\tilde{\mathbf{W}}_{\mathbf{m}} \Rightarrow \tilde{\mathbf{W}}_{\mathbf{m}} + \tilde{\mathbf{F}} + \hat{\vec{\nabla}}\tilde{\mathbf{E}}, \tag{10.73}$$

where $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{E}}$ are arbitrary sedeons satisfy the following conditions:

$$\hat{\vec{\nabla}}\tilde{\mathbf{F}} = 0, \tag{10.74}$$

$$\hat{\vec{\nabla}}\hat{\vec{\nabla}}\tilde{\mathbf{E}} = 0. \tag{10.75}$$

The condition (10.74) indicates that the potential $\tilde{\mathbf{W}}_{\mathbf{m}}$ is defined up to an additive function $\tilde{\mathbf{F}}$ satisfying the homogeneous first-order wave equation, while expression (10.75) means that $\tilde{\mathbf{E}}$ satisfy the homogeneous second-order wave equation. Let us consider the generalized gradient gauge condition. For the potential determined by the expression (10.7) the function $\tilde{\mathbf{E}}$ can be chosen as follows:

$$\tilde{\mathbf{E}} = \varepsilon_1 - i\varepsilon_2\mathbf{e_{tr}} - i\varepsilon_3\mathbf{e_t} + i\varepsilon_4\mathbf{e_r} + \vec{E}_1\mathbf{e_{tr}} - i\vec{E}_2 + \vec{E}_3\mathbf{e_r} + \vec{E}_4\mathbf{e_t}, \tag{10.76}$$

where components ε_S and $\vec{E}_S$ are arbitrary real functions of coordinates and time. Then the replacement (10.73) leads us to the following system of substitutions:

$$\begin{aligned}
a_1 &\Rightarrow a_1 + \partial\varepsilon_1 - \left(\vec{\nabla}\cdot\vec{E}_1\right) - m\varepsilon_4,\\
a_2 &\Rightarrow a_2 + \partial\varepsilon_2 - \left(\vec{\nabla}\cdot\vec{E}_2\right) + m\varepsilon_3,\\
a_3 &\Rightarrow a_3 + \partial\varepsilon_3 - \left(\vec{\nabla}\cdot\vec{E}_3\right) - m\varepsilon_2,\\
a_4 &\Rightarrow a_4 + \partial\varepsilon_4 - \left(\vec{\nabla}\cdot\vec{E}_4\right) + m\varepsilon_1,\\
\vec{A}_1 &\Rightarrow \vec{A}_1 + \partial\vec{E}_1 - \vec{\nabla}\varepsilon_1 + i\left[\vec{\nabla}\times\vec{E}_2\right] + m\vec{E}_4,\\
\vec{A}_2 &\Rightarrow \vec{A}_2 + \partial\vec{E}_2 - \vec{\nabla}\varepsilon_2 - i\left[\vec{\nabla}\times\vec{E}_1\right] - m\vec{E}_3,\\
\vec{A}_3 &\Rightarrow \vec{A}_3 + \partial\vec{E}_3 - \vec{\nabla}\varepsilon_3 - i\left[\vec{\nabla}\times\vec{E}_4\right] + m\vec{E}_2,\\
\vec{A}_4 &\Rightarrow \vec{A}_4 + \partial\vec{E}_4 - \vec{\nabla}\varepsilon_4 + i\left[\vec{\nabla}\times\vec{E}_3\right] - m\vec{E}_1.
\end{aligned} \tag{10.77}$$

If we chose the vector part equal to zero ($\vec{E}_S = 0$), then the substitutions (10.77) are reduced to (10.11) and to (10.33) for the zero mass quantum. Analogous substitutions for the field strengths have the following form:

$$
\begin{aligned}
&g_1 \Rightarrow g_1 + \partial\varepsilon_1 + \left(\vec{\nabla}\cdot\vec{E}_1\right) - m\varepsilon_4,\\
&g_2 \Rightarrow g_2 - \partial\varepsilon_2 - \left(\vec{\nabla}\cdot\vec{E}_2\right) - m\varepsilon_3,\\
&g_3 \Rightarrow g_3 + \partial\varepsilon_3 - \left(\vec{\nabla}\cdot\vec{E}_3\right) - m\varepsilon_2,\\
&g_3 \Rightarrow g_3 - \partial\varepsilon_4 + \left(\vec{\nabla}\cdot\vec{E}_4\right) - m\varepsilon_1,\\
&\vec{G}_1 \Rightarrow \vec{G}_1 - \partial\vec{E}_1 - \vec{\nabla}\varepsilon_1 + i\left[\vec{\nabla}\times\vec{E}_2\right] + m\vec{E}_4,\\
&\vec{G}_2 \Rightarrow \vec{G}_2 + \partial\vec{E}_2 + \vec{\nabla}\varepsilon_2 + i\left[\vec{\nabla}\times\vec{E}_1\right] + m\vec{E}_3,\\
&\vec{G}_3 \Rightarrow \vec{G}_3 + \partial\vec{E}_3 - \vec{\nabla}\varepsilon_3 + i\left[\vec{\nabla}\times\vec{E}_4\right] - m\vec{E}_2,\\
&\vec{G}_4 \Rightarrow \vec{G}_4 - \partial\vec{E}_4 + \vec{\nabla}\varepsilon_4 + i\left[\vec{\nabla}\times\vec{E}_3\right] - m\vec{E}_1.
\end{aligned}
\qquad (10.78)
$$

If we chose the vector part equal to zero, then the substitutions (10.78) are reduced to (10.15) and to (10.37) and (10.42) for the zero mass quantum.

10.6. Conclusion

Thus we have presented the sypersymmetric scalar-vector equations for massive and massless fields. The gauge invariance for the potentials described by second-order and first-order wave equations and for the field strengths described by the systems of Maxwell-like equations has been demonstrated.

Application 1. Matrix representation of sedeons

Let us consider a matrix representation of the sedeon. In general, the sedeon is equivalent to the 16 × 16 matrix. Working with such a matrix is extremely difficult because of its high dimensionality. However, this matrix cab be represented in the compact form of 4 × 4 block matrices. Let us consider the sedeon $\tilde{\mathbf{V}}$ in the basis $\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$:

$$\tilde{\mathbf{V}} = \mathbf{e_0}\overline{\mathbf{V}}_0 + \mathbf{e_1}\overline{\mathbf{V}}_1 + \mathbf{e_2}\overline{\mathbf{V}}_2 + \mathbf{e_3}\overline{\mathbf{V}}_3 . \qquad \text{(A 1.1)}$$

The sedeonic product of $\mathbf{e_1}$ and $\tilde{\mathbf{V}}$ can be written as

$$\mathbf{e_1}\tilde{\mathbf{V}} = \mathbf{e_0}\overline{\mathbf{V}}_1 + \mathbf{e_1}\overline{\mathbf{V}}_0 - i\mathbf{e_2}\overline{\mathbf{V}}_3 + i\mathbf{e_3}\overline{\mathbf{V}}_2 , \qquad \text{(A 1.1)}$$

therefore the sedeonic unit $\mathbf{e_1}$ enables the following matrix representation:

$$\mathbf{e_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} . \qquad \text{(A 1.3)}$$

Analogously:

$$\mathbf{e_0} = 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \qquad \text{(A 1.4)}$$

Thus, using (A 1.3) and (A 1.4), we can write a sedeon $\tilde{\mathbf{V}}$ (in $\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ basis) in the following matrix form:

$$\tilde{\mathbf{V}} = \begin{pmatrix} \overline{\mathbf{V}}_0 & \overline{\mathbf{V}}_1 & \overline{\mathbf{V}}_2 & \overline{\mathbf{V}}_3 \\ \overline{\mathbf{V}}_1 & \overline{\mathbf{V}}_0 & -i\overline{\mathbf{V}}_3 & i\overline{\mathbf{V}}_2 \\ \overline{\mathbf{V}}_2 & i\overline{\mathbf{V}}_3 & \overline{\mathbf{V}}_0 & -i\overline{\mathbf{V}}_1 \\ \overline{\mathbf{V}}_3 & -i\overline{\mathbf{V}}V_2 & i\overline{\mathbf{V}}_1 & \overline{\mathbf{V}}_0 \end{pmatrix} . \qquad \text{(A 1.5)}$$

On the other hand we can write sedeon $\tilde{\mathbf{V}}$ using $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ basis in the following scalar-vector form:

$$\tilde{\mathbf{V}} = \mathbf{V}_0\mathbf{a_0} + \mathbf{V}_1\mathbf{a_1} + \mathbf{V}_2\mathbf{a_2} + \mathbf{V}_3\mathbf{a_3} \,. \tag{A 1.6}$$

Then the basis elements $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ have the following matrix representation:

$$\mathbf{a_0} = 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{a_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\mathbf{a_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \mathbf{a_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A 1.7}$$

Using (A 1.7) a sedeon $\tilde{\mathbf{V}}$ can be written in $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ basis as 4×4 block matrix:

$$\tilde{\mathbf{V}} = \begin{pmatrix} \mathbf{V}_0 & \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 \\ \mathbf{V}_1 & \mathbf{V}_0 & -i\mathbf{V}_3 & i\mathbf{V}_2 \\ \mathbf{V}_2 & i\mathbf{V}_3 & \mathbf{V}_0 & -i\mathbf{V}_1 \\ \mathbf{V}_3 & -i\mathbf{V}_2 & i\mathbf{V}_1 & \mathbf{V}_0 \end{pmatrix}. \tag{A 1.8}$$

Thus the sixteen-component sedeon can be written as a 16×16 matrix, which can be represented in two different compact 4×4 form. First representation in $\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ basis is (A 1.5) with $\bar{\mathbf{V}}_\alpha$ components in $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ basis

$$\bar{\mathbf{V}}_\alpha = \begin{pmatrix} V_{\alpha 0} & V_{\alpha 1} & V_{\alpha 2} & V_{\alpha 3} \\ V_{\alpha 1} & V_{\alpha 0} & -iV_{\alpha 3} & iV_{\alpha 2} \\ V_{\alpha 2} & iV_{\alpha 3} & V_{\alpha 0} & -iV_{\alpha 1} \\ V_{\alpha 3} & -iV_{\alpha 2} & iV_{\alpha 1} & V_{\alpha 0} \end{pmatrix}. \tag{A 1.9}$$

Second representation in $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ basis is (A 1.8) with $\mathbf{V}_\beta$ components in $\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ basis

$$\mathbf{V}_\beta = \begin{pmatrix} V_{0\beta} & V_{1\beta} & V_{2\beta} & V_{3\beta} \\ V_{1\beta} & V_{0\beta} & -iV_{3\beta} & iV_{2\beta} \\ V_{2\beta} & iV_{3\beta} & V_{0\beta} & -iV_{1\beta} \\ V_{3\beta} & -iV_{2\beta} & iV_{1\beta} & V_{0\beta} \end{pmatrix}. \quad \text{(A 1.10)}$$

Let us consider the relations between unit vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ and Dirac matrices. Introducing new values

$$\mathbf{W}_1 = \frac{1}{2}(\mathbf{V}_0 + \mathbf{V}_3), \ \mathbf{W}_2 = \frac{1}{2}(\mathbf{V}_1 + i\mathbf{V}_2),$$
$$\mathbf{W}_3 = \frac{1}{2}(\mathbf{V}_1 - i\mathbf{V}_2), \ \mathbf{W}_4 = \frac{1}{2}(\mathbf{V}_0 - \mathbf{V}_3), \quad \text{(A 1.11)}$$

we can write the sedeon (A 1.6) in the basis of eigenfunctions of operator $\mathbf{a_3}$ in the following form:

$$\tilde{\mathbf{V}} = \mathbf{W}_1(1+\mathbf{a_3}) + \mathbf{W}_2(\mathbf{a_1} - i\mathbf{a_2}) + \mathbf{W}_3(\mathbf{a_1} + i\mathbf{a_2}) + \mathbf{W}_4(1-\mathbf{a_3}), \quad \text{(A 1.12)}$$

where set of values

$$(1+\mathbf{a_3}), \ (\mathbf{a_1} - i\mathbf{a_2}), \ (\mathbf{a_1} + i\mathbf{a_2}), \ (1-\mathbf{a_3}) \quad \text{(A 1.13)}$$

is the new sedeonic basis. Then the action of vector operators $\mathbf{a_m}$ can be represented as

$$\mathbf{a_1}\tilde{\mathbf{V}} = \mathbf{W}_2(1+\mathbf{a_3}) + \mathbf{W}_1(\mathbf{a_1} - i\mathbf{a_2}) + \mathbf{W}_4(\mathbf{a_1} + i\mathbf{a_2}) + \mathbf{W}_3(1-\mathbf{a_3}),$$

$$\mathbf{a_2}\tilde{\mathbf{V}} = -i\mathbf{W}_2(1+\mathbf{a_3}) + i\mathbf{W}_1(\mathbf{a_1} - i\mathbf{a_2}) - i\mathbf{W}_4(\mathbf{a_1} + i\mathbf{a_2}) + i\mathbf{W}_3(1-\mathbf{a_3}), \quad \text{(A 1.14)}$$

$$\mathbf{a_3}\tilde{\mathbf{V}} = \mathbf{W}_1(1+\mathbf{a_3}) - \mathbf{W}_2(\mathbf{a_1} - i\mathbf{a_2}) + \mathbf{W}_3(\mathbf{a_1} + i\mathbf{a_2}) - \mathbf{W}_4(1-\mathbf{a_3}).$$

Therefore the unit vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ can be written in the new basis as the following 4 × 4 matrices:

$$\mathbf{a_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \mathbf{a_2} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \ \mathbf{a_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{(A 1.15)}$$

which coincide with spin operators $\hat{\sigma}_m$ in Dirac theory [27]:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad \text{(A 1.16)}$$

Thus, the matrix operators $\mathbf{e}_\alpha$ and $\mathbf{a}_\beta$, can be presented as 16×16 matrices. The 4×4 matrix presentation is valid only for specified bases and only in case when operators $\mathbf{e}_\alpha$ and $\mathbf{a}_\beta$ act separately and independently.

Application 2. Space-time sedenions

The well known sixteen-component hypercomplex numbers, sedenions, are obtained from octonions by the Cayley-Dickson extension procedure [51]. In this case the sedenion is defined as

$$S = O_1 + O_2 \boldsymbol{e} , \quad \text{(A 2.1)}$$

where O_i is an octonion and the parameter of duplication $\boldsymbol{e}$ is similar to imaginary unit ($\boldsymbol{e}^2 = -1$). The algebra of sedenions has the specific rules of multiplication. The product of two sedenions

$$S_1 = O_{11} + O_{12}\boldsymbol{e} ,$$
$$S_2 = O_{21} + O_{22}\boldsymbol{e}$$

is defined as

$$S_1 S_2 = (O_{11} + O_{12}\boldsymbol{e})(O_{21} + O_{22}\boldsymbol{e}) = \left(O_{11}O_{21} - \bar{O}_{22}O_{12}\right) + \left(O_{22}O_{11} + O_{12}\bar{O}_{21}\right)\boldsymbol{e} , \quad \text{(A 2.2)}$$

where $\bar{O}_i$ is conjugated octonion. The sedenionic multiplication (A 2.2) allows one to introduce a well defined norm of sedenion. However such procedure of constructing the higher hypercomplex numbers leads to the fact that the sedenions as well as octonions generate normed but nonassociative algebra [4]. This greatly complicates the use of the Cayley-Dickson sedenions in the physical applications.

In this section we present an alternative version of the associative sixteen-component hypercomplex numbers named "space-time sedenions" [52] and demonstrate some of its application to the generalization of the field theory equations.

П 2.1. Sedenionic space-time algebra

It is known, the quaternion is a four-component object, which can be presented in the following form:

$$\hat{q} = q_0 \mathbf{a}_0 + q_1 \mathbf{a}_1 + q_2 \mathbf{a}_2 + q_3 \mathbf{a}_3 , \quad \text{(A 2.3)}$$

where components $q_\alpha\,(\alpha = 0, 1, 2, 3)$ are numbers (complex in general), $\mathbf{a}_0 \equiv 1$ is scalar units and values $\mathbf{a}_\mathrm{m}$ $(\mathrm{m} = 1, 2, 3)$ are quaternionic units, which are interpreted as unit vectors. The rules of multiplication and commutation for $\mathbf{a}_\mathrm{m}$ are presented in Table 4. We introduce also the space-time basis $\mathbf{e}_0, \mathbf{e}_\mathbf{t}, \mathbf{e}_\mathbf{r}, \mathbf{e}_\mathbf{tr}$, which is responsible for the space-time inversions. The indexes **t** and **r** indicate the transformations (**t** for time inversion and **r** for spatial inversion), which change the corresponding values. The value $\mathbf{e}_0 \equiv 1$ is a absolute scalar unit. For convenience we introduce numerical designations $\mathbf{e}_1 \equiv \mathbf{e}_\mathbf{t}$ is time scalar unit; $\mathbf{e}_2 \equiv \mathbf{e}_\mathbf{r}$ is space scalar unit ; $\mathbf{e}_3 \equiv \mathbf{e}_\mathbf{tr}$ is space-time scalar unit. The rules of multiplication and commutation for this basis $\mathbf{e}_\mathrm{m}$ we choose similar to the rules for quaternionic units (see Table 5).

Table 4.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	-1	$\mathbf{a}_3$	$-\mathbf{a}_2$
$\mathbf{a}_2$	$-\mathbf{a}_3$	-1	$\mathbf{a}_1$
$\mathbf{a}_3$	$\mathbf{a}_2$	$-\mathbf{a}_1$	-1

Table 5.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{e}_1$
$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1

Note that the unit vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and the space-time units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ generate the anticommutative algebras:

$$\begin{aligned} \mathbf{a}_\mathbf{n}\mathbf{a}_\mathbf{m} &= -\mathbf{a}_\mathbf{m}\mathbf{a}_\mathbf{n}, \\ \mathbf{e}_\mathbf{n}\mathbf{e}_\mathbf{m} &= -\mathbf{e}_\mathbf{m}\mathbf{e}_\mathbf{n}, \end{aligned} \tag{A 2.4}$$

for $\mathbf{n} \neq \mathbf{m}$, but basis elements $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ commute with elements $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$:

$$\mathbf{e}_\mathbf{n}\mathbf{a}_\mathbf{m} = \mathbf{a}_\mathbf{m}\mathbf{e}_\mathbf{n}, \tag{A 2.5}$$

for any $\mathbf{n}$ и $\mathbf{m}$. Then we can introduce the sixteen-component space-time sedenion $\tilde{V}$ in the following form:

$$\tilde{V} = \mathbf{e_0}\left(V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3}\right)$$
$$+ \mathbf{e_1}\left(V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3}\right)$$
$$+ \mathbf{e_2}\left(V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3}\right) \qquad \text{(A 2.6)}$$
$$+ \mathbf{e_3}\left(V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3}\right).$$

The sedenionic components $V_{\alpha\beta}$ are numbers (complex in general). Introducing designation of scalar and vector values in accordance with the following relations:

$$\begin{aligned}
V &= \mathbf{e_0}V_{00}\mathbf{a_0}, \\
\vec{V} &= \mathbf{e_0}\left(V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3}\right), \\
V_{\mathbf{t}} \equiv V_{\mathbf{1}} &= \mathbf{e_1}V_{10}\mathbf{a_0}, \\
\vec{V}_{\mathbf{t}} \equiv \vec{V}_{\mathbf{1}} &= \mathbf{e_1}\left(V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3}\right), \\
V_{\mathbf{r}} \equiv V_{\mathbf{2}} &= \mathbf{e_2}V_{20}\mathbf{a_0}, \\
\vec{V}_{\mathbf{r}} \equiv \vec{V}_{\mathbf{2}} &= \mathbf{e_2}\left(V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3}\right), \\
V_{\mathbf{tr}} \equiv V_{\mathbf{3}} &= \mathbf{e_3}V_{30}\mathbf{a_0}, \\
\vec{V}_{\mathbf{tr}} \equiv \vec{V}_{\mathbf{3}} &= \mathbf{e_3}\left(V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3}\right).
\end{aligned} \qquad \text{(A 2.7)}$$

Then we can represent the sedenion in the following scalar-vector form:

$$\tilde{V} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}}. \qquad \text{(A 2.8)}$$

Thus, the sedenionic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

- Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
- Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
- Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
- Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Further we will use the symbol 1 instead units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity. Introducing the designations of scalar-vector values

$$\begin{aligned}
\bar{\boldsymbol{V}}_0 &= V_{00} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3}\,, \\
\bar{\boldsymbol{V}}_1 &= V_{10} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3}\,, \\
\bar{\boldsymbol{V}}_2 &= V_{20} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3}\,, \\
\bar{\boldsymbol{V}}_3 &= V_{30} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3}\,.
\end{aligned} \tag{A 2.9}$$

Then we can write the sedenion using space-time basis in the following compact form:

$$\tilde{\boldsymbol{V}} = \bar{\boldsymbol{V}}_0 + \mathbf{e_1}\bar{\boldsymbol{V}}_1 + \mathbf{e_2}\bar{\boldsymbol{V}}_2 + \mathbf{e_3}\bar{\boldsymbol{V}}_3\,. \tag{A 2.10}$$

On the other hand, introducing the designations of space-time sedenion-scalars

$$\begin{aligned}
\boldsymbol{V}_0 &= (V_{00} + \mathbf{e_1}V_{10} + \mathbf{e_2}V_{20} + \mathbf{e_3}V_{30})\,, \\
\boldsymbol{V}_1 &= (V_{01} + \mathbf{e_1}V_{11} + \mathbf{e_2}V_{21} + \mathbf{e_3}V_{31})\,, \\
\boldsymbol{V}_2 &= (V_{02} + \mathbf{e_1}V_{12} + \mathbf{e_2}V_{22} + \mathbf{e_3}V_{32})\,, \\
\boldsymbol{V}_3 &= (V_{03} + \mathbf{e_1}V_{13} + \mathbf{e_2}V_{23} + \mathbf{e_3}V_{33})
\end{aligned} \tag{A 2.11}$$

we can write the sedenion in vector basis as

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}_0 + \boldsymbol{V}_1\mathbf{a_1} + \boldsymbol{V}_2\mathbf{a_2} + \boldsymbol{V}_3\mathbf{a_3}\,, \tag{A 2.12}$$

or introducing the sedenion-vector

$$\vec{\boldsymbol{V}} = \vec{V} + \vec{V}_{\mathbf{t}} + \vec{V}_{\mathbf{r}} + \vec{V}_{\mathbf{tr}} = \boldsymbol{V}_1\mathbf{a_1} + \boldsymbol{V}_2\mathbf{a_2} + \boldsymbol{V}_3\mathbf{a_3}\,, \tag{A 2.13}$$

we can rewrite the sedenion in following compact form:

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}_0 + \vec{\boldsymbol{V}}\,. \tag{A 2.14}$$

Further we will indicate sedenion-scalars and sedenion-vectors with the bold capital letters.

Let us consider the sedenionic multiplication in detail. The sedenionic product of two sedenions $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ can be represented in the following form:

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} = \left(\boldsymbol{A}_0 + \vec{\boldsymbol{A}}\right)\left(\boldsymbol{B}_0 + \vec{\boldsymbol{B}}\right) = \boldsymbol{A}_0\boldsymbol{B}_0 + \boldsymbol{A}_0\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}\boldsymbol{B}_0 + \left(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}\right) + \left[\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}\right]. \tag{A 2.15}$$

Here we denoted the sedenionic scalar multiplication of two sedenion-vectors (internal product) by symbol “ $\cdot$ ” and round brackets:

$$\left(\vec{A}\cdot\vec{B}\right) = -A_1B_1 - A_2B_2 - A_3B_3, \qquad \text{(A 2.16)}$$

and sedenionic vector multiplication (external product) by symbol "×" and square brackets:

$$\left[\vec{A}\times\vec{B}\right] = \left(A_2B_3 - A_3B_2\right)\mathbf{a_1} + \left(A_3B_1 - A_1B_3\right)\mathbf{a_2} + \left(A_1B_2 - A_2B_1\right)\mathbf{a_3}. \qquad \text{(A 2.17)}$$

Thus the sedenionic product

$$\tilde{F} = \tilde{A}\tilde{B} = F_0 + \vec{F} \qquad \text{(A 2.18)}$$

has the following components:

$$\begin{aligned} F_0 &= A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3, \\ F_1 &= A_1B_0 + A_0B_1 + \left(A_2B_3 - A_3B_2\right), \\ F_2 &= A_2B_0 + A_0B_2 + \left(A_3B_1 - A_1B_3\right), \\ F_3 &= A_3B_0 + A_0B_3 + \left(A_1B_2 - A_2B_1\right). \end{aligned} \qquad \text{(A 2.19)}$$

Note that in the sedenionic algebra the square of vector is defined as

$$\vec{A}^2 = \left(\vec{A}\cdot\vec{A}\right) = -A_1^2 - A_2^2 - A_3^2. \qquad \text{(A 2.20)}$$

On the other hand, the square of modulus of vector is

$$\left|\vec{A}\right|^2 = -\left(\vec{A}\cdot\vec{A}\right) = A_1^2 + A_2^2 + A_3^2. \qquad \text{(A 2.21)}$$

It is positively defined value.

A 2.2. Sedenionic spatial rotation and space-time conjugation

The rotation of sedenion $\tilde{V}$ on the angle θ around the absolute unit vector $\vec{n}$ is realized by uncompleted sedenion

$$\tilde{U} = \cos(\theta/2) + \vec{n}\sin(\theta/2) \qquad \text{(A 2.22)}$$

and by conjugated sedenion:

$$\tilde{U}^* = \cos(\theta/2) - \vec{n}\sin(\theta/2). \qquad \text{(A 2.23)}$$

They satisfy the following relation:

$$\tilde{\boldsymbol{U}}\tilde{\boldsymbol{U}}^* = \tilde{\boldsymbol{U}}^*\tilde{\boldsymbol{U}} = 1 . \qquad \text{(A 2.24)}$$

The transformed sedenion $\tilde{\boldsymbol{V}}'$ is defined as sedenionic product:

$$\tilde{\boldsymbol{V}}' = \tilde{\boldsymbol{U}}^* \tilde{\boldsymbol{V}} \tilde{\boldsymbol{U}} \quad . \qquad \text{(A 2.25)}$$

Thus the transformed sedenion $\tilde{\boldsymbol{V}}'$ can be written as

$$\tilde{\boldsymbol{V}}' = \left[\cos(\theta/2) - \vec{n}\sin(\theta/2)\right]\left(\boldsymbol{V}_0 + \vec{\boldsymbol{V}}\right)\left[\cos(\theta/2) + \vec{n}\sin(\theta/2)\right]$$
$$= \boldsymbol{V}_0 + \vec{\boldsymbol{V}}\cos\theta - \vec{n}\left(\vec{n}\cdot\vec{\boldsymbol{V}}\right)(1-\cos\theta) - \left[\vec{n}\times\vec{\boldsymbol{V}}\right]\sin\theta . \qquad \text{(A 2.26)}$$

It is clearly seen that rotation does not transform the sedenion-scalar part, but the sedenionic vector $\vec{\boldsymbol{V}}$ is rotated on the angle θ around $\vec{n}$.

The operations of time inversion ($\hat{R}_{\mathbf{t}}$), space inversion ($\hat{R}_{\mathbf{r}}$) and space-time inversion ($\hat{R}_{\mathbf{tr}}$) are connected with transformations in $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ basis and can be presented as

$$\hat{R}_{\mathbf{t}}\tilde{\boldsymbol{V}} = -\mathbf{e}_2\tilde{\boldsymbol{V}}\mathbf{e}_2 = \overline{\boldsymbol{V}}_0 - \mathbf{e}_1\overline{\boldsymbol{V}}_1 + \mathbf{e}_2\overline{\boldsymbol{V}}_2 - \mathbf{e}_3\overline{\boldsymbol{V}}_3 ,$$
$$\hat{R}_{\mathbf{r}}\tilde{\boldsymbol{V}} = -\mathbf{e}_1\tilde{\boldsymbol{V}}\mathbf{e}_1 = \overline{\boldsymbol{V}}_0 + \mathbf{e}_1\overline{\boldsymbol{V}}_1 - \mathbf{e}_2\overline{\boldsymbol{V}}_2 - \mathbf{e}_3\overline{\boldsymbol{V}}_3 , \qquad \text{(A 2.27)}$$
$$\hat{R}_{\mathbf{tr}}\tilde{\boldsymbol{V}} = -\mathbf{e}_3\tilde{\boldsymbol{V}}\mathbf{e}_3 = \overline{\boldsymbol{V}}_0 - \mathbf{e}_1\overline{\boldsymbol{V}}_1 - \mathbf{e}_2\overline{\boldsymbol{V}}_2 + \mathbf{e}_3\overline{\boldsymbol{V}}_3 .$$

A 2.3. Sedenionic Lorentz transformations

In sedenionic algebra the relativistic event four-vector can be represented in the follow sedenionic form:

$$\tilde{\boldsymbol{S}} = \mathbf{e}_{\mathbf{t}}ct + \mathbf{e}_{\mathbf{r}}\vec{r} . \qquad \text{(A 2.28)}$$

The square of this value is the Lorentz invariant

$$\tilde{\boldsymbol{S}}\,\tilde{\boldsymbol{S}} = -c^2t^2 + x^2 + y^2 + z^2 . \qquad \text{(A 2.29)}$$

The Lorentz transformation of event four-vector is realized by uncompleted sedenions

$$\tilde{\boldsymbol{L}} = \cosh\vartheta + \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta , \qquad \text{(A 2.30)}$$
$$\tilde{\boldsymbol{L}}^* = \cosh\vartheta - \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta , \qquad \text{(A 2.31)}$$

where $\tanh 2\vartheta = \mathrm{v}/c$, v is a speed of motion along the absolute unit vector $\vec{m}$. Note that

$$\tilde{\boldsymbol{L}}^{*}\tilde{\boldsymbol{L}} = \tilde{\boldsymbol{L}}\,\tilde{\boldsymbol{L}}^{*} = 1. \tag{A 2.32}$$

For example, the transformed event four-vector $\tilde{\boldsymbol{S}}'$ is written as

$$\begin{aligned}\tilde{\boldsymbol{S}}' = \tilde{\boldsymbol{L}}^{*}\,\tilde{\boldsymbol{S}}\,\tilde{\boldsymbol{L}} &= \left(\cosh\vartheta - \mathbf{e}_{\mathbf{tr}}\sinh\vartheta\,\vec{m}\right)\left(\mathbf{e}_{\mathbf{t}}ct + \mathbf{e}_{\mathbf{r}}\vec{r}\right)\left(\cosh\vartheta + \mathbf{e}_{\mathbf{tr}}\sinh\vartheta\,\vec{m}\right)\\ &= \mathbf{e}_{\mathbf{t}}ct\,\cosh 2\vartheta + \mathbf{e}_{\mathbf{t}}\left(\vec{m}\cdot\vec{r}\right)\sinh 2\vartheta \\ &+\mathbf{e}_{\mathbf{r}}\vec{r} - \mathbf{e}_{\mathbf{r}}ct\vec{m}\,\sinh 2\vartheta + \mathbf{e}_{\mathbf{r}}\left(\vec{m}\cdot\vec{r}\right)\vec{m}\left(1-\cosh 2\vartheta\right)\ .\end{aligned} \tag{A 2.33}$$

Separating in (A 2.33) the values with $\mathbf{e}_{\mathbf{t}}$ and $\mathbf{e}_{\mathbf{r}}$ we get the well known formulas for time and coordinates transformation [26]:

$$t' = \frac{t - x\,v/c^2}{\sqrt{1-v^2/c^2}}\ ,\ x' = \frac{x - t\,v}{\sqrt{1-\mathrm{v}^2/c^2}}\ ,\ y' = y\ ,\ z' = z\ , \tag{A 2.34}$$

where x is the coordinate along the $\vec{m}$ vector.

Let us also consider the Lorentz transformation of the full sedenion $\tilde{\boldsymbol{V}}$. The Lorentz transformation for any sedenion $\tilde{\boldsymbol{V}}'$ can be written as sedenionic product

$$\tilde{\boldsymbol{V}}' = \tilde{\boldsymbol{L}}^{*}\,\tilde{\boldsymbol{V}}\,\tilde{\boldsymbol{L}}\ \ . \tag{A 2.35}$$

The transformed sedenion has the following components:

$$\begin{aligned}&V' = V\ ,\\ &V'_{\mathbf{tr}} = V_{\mathbf{tr}}\ ,\\ &V'_{\mathbf{r}} = V_{\mathbf{r}}\cosh 2\vartheta - \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\sinh 2\vartheta\ ,\\ &V'_{\mathbf{t}} = V_{\mathbf{t}}\cosh 2\vartheta - \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\sinh 2\vartheta\ ,\\ &\vec{V}' = \vec{V}\cosh 2\vartheta + \left(\vec{m}\cdot\vec{V}\right)\vec{m}\left(1-\cosh 2\vartheta\right) - \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}_{\mathbf{rt}}\right]\sinh 2\vartheta\ ,\\ &\vec{V}'_{\mathbf{tr}} = \vec{V}_{\mathbf{tr}}\cosh 2\vartheta + \left(\vec{m}\cdot\vec{V}_{\mathbf{tr}}\right)\vec{m}\left(1-\cosh 2\vartheta\right) - \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}\right]\sinh 2\vartheta\ ,\\ &\vec{V}'_{\mathbf{r}} = \vec{V}_{\mathbf{r}} - \left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\vec{m}\left(1-\cosh 2\vartheta\right) - \mathbf{e}_{\mathbf{tr}}V_{\mathbf{t}}\ \vec{m}\sinh 2\vartheta\ ,\\ &\vec{V}'_{\mathbf{t}} = \vec{V}_{\mathbf{t}} - \left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\vec{m}\left(1-\cosh 2\vartheta\right) - \mathbf{e}_{\mathbf{tr}}V_{\mathbf{r}}\ \vec{m}\sinh 2\vartheta\ .\end{aligned} \tag{A 2.36}$$

A 2.4. Subalgebras of space-time quaternions and octonions

The sedenionic basis introduced above enables constructing different types of low-dimensional hypercomplex numbers. For example one can introduce space-time complex numbers

$$Z_{\mathbf{t}} = z_1 + \mathbf{e}_{\mathbf{t}} z_2 , \tag{A 2.37}$$

$$Z_{\mathbf{r}} = z_1 + \mathbf{e}_{\mathbf{r}} z_2 , \tag{A 2.38}$$

$$Z_{\mathbf{tr}} = z_1 + \mathbf{e}_{\mathbf{tr}} z_2 , \tag{A 2.39}$$

which are transformed under space and time conjugation. Moreover we can consider the space-time quaternions, which differ in their properties with respect to the operations of the spatial and time inversion

$$\widehat{q} = q_0 \mathbf{a_0} + \mathbf{e_0} \left(q_1 \mathbf{a_1} + q_2 \mathbf{a_2} + q_3 \mathbf{a_3} \right) , \tag{A 2.40}$$

$$\widehat{q}_{\mathbf{t}} = q_0 \mathbf{a_0} + \mathbf{e}_{\mathbf{t}} \left(q_1 \mathbf{a_1} + q_2 \mathbf{a_2} + q_3 \mathbf{a_3} \right) , \tag{A 2.41}$$

$$\widehat{q}_{\mathbf{r}} = q_0 \mathbf{a_0} + \mathbf{e}_{\mathbf{r}} \left(q_1 \mathbf{a_1} + q_2 \mathbf{a_2} + q_3 \mathbf{a_3} \right) , \tag{A 2.42}$$

$$\widehat{q}_{\mathbf{tr}} = q_0 \mathbf{a_0} + \mathbf{e}_{\mathbf{tr}} \left(q_1 \mathbf{a_1} + q_2 \mathbf{a_2} + q_3 \mathbf{a_3} \right) . \tag{A 2.43}$$

The absolute quaternion (A 2.40) is the sum of the absolute scalar and absolute vector. It remains constant under the transformations of space and time inversion (A 2.27). Time quaternion $\widehat{q}_{\mathbf{t}}$, space quaternion $\widehat{q}_{\mathbf{r}}$ and space-time quaternion $\widehat{q}_{\mathbf{tr}}$ are transformed under inversions in accordance with the commutation rules for the basis elements $\mathbf{e}_{\mathbf{t}}$, $\mathbf{e}_{\mathbf{r}}$, $\mathbf{e}_{\mathbf{tr}}$. For example, performing the operation of time inversion with the quaternion $\widehat{q}_{\mathbf{t}}$ we obtain the conjugated quaternion

$$\hat{R}_{\mathbf{t}} \widehat{q}_{\mathbf{t}} = -\mathbf{e}_{\mathbf{r}} \widehat{q}_{\mathbf{t}} \mathbf{e}_{\mathbf{r}} = q_0 \mathbf{a_0} - \mathbf{e}_{\mathbf{t}} \left(q_1 \mathbf{a_1} + q_2 \mathbf{a_2} + q_3 \mathbf{a_3} \right) . \tag{A 2.44}$$

Moreover, the sedenionic basis allows one to construct various types of space-time eight-component octonions:

$$\breve{G}_{\mathbf{t}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_t}G_{10} + \mathbf{e_t}\left(G_{11}\mathbf{a_1} + G_{12}\mathbf{a_2} + G_{13}\mathbf{a_3}\right), \quad \text{(A 2.45)}$$

$$\breve{G}_{\mathbf{r}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_r}G_{20} + \mathbf{e_r}\left(G_{21}\mathbf{a_1} + G_{22}\mathbf{a_2} + G_{23}\mathbf{a_3}\right), \quad \text{(A 2.46)}$$

$$\breve{G}_{\mathbf{tr}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_{tr}}G_{30} + \mathbf{e_{tr}}\left(G_{31}\mathbf{a_1} + G_{32}\mathbf{a_2} + G_{33}\mathbf{a_3}\right). \quad \text{(A 2.47)}$$

A 2.5. Sedenionic equations of relativistic quantum mechanics

In sedenionic algebra the Einstein relation for energy and momentum

$$E^2 - c^2 p^2 - m_0^2 c^4 = 0 \quad \text{(A 2.48)}$$

can be presented in the following form:

$$\left(\mathbf{e_t}E + \mathbf{e_r}c\vec{p} + i\mathbf{e_{tr}}m_0c^2\right)\left(\mathbf{e_t}E + \mathbf{e_r}c\vec{p} + i\mathbf{e_{tr}}m_0c^2\right) = 0\,. \quad \text{(A 2.49)}$$

Changing classical energy E and momentum $\vec{p}$ on corresponding quantum-mechanical operators:

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \quad \text{и} \quad \hat{\vec{p}} = -i\hbar\vec{\nabla}\,, \quad \text{(A 2.50)}$$

we get the sedenionic wave equation for relativistic particle:

$$\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} + \mathbf{e_{tr}}\frac{m_0c}{\hbar}\right)\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_r}\vec{\nabla} + \mathbf{e_{tr}}\frac{m_0c}{\hbar}\right)\tilde{\boldsymbol{\psi}} = 0\,, \quad \text{(A 2.51)}$$

where the wave function is sedenion

$$\tilde{\boldsymbol{\psi}}(t,\vec{r}) = \boldsymbol{\psi}_0(t,\vec{r}) + \vec{\boldsymbol{\psi}}(t,\vec{r})\,. \quad \text{(A 2.52)}$$

Note that for electrically charged particle in an external electromagnetic field we have the following sedenionic wave equation:

$$\begin{gathered}\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{e_t}\frac{ie}{\hbar c}\varphi - \mathbf{e_r}\vec{\nabla} + \mathbf{e_r}\frac{ie}{\hbar c}\vec{A} + \mathbf{e_{tr}}\frac{m_0c}{\hbar}\right)\\ \times\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{e_t}\frac{ie}{\hbar c}\varphi - \mathbf{e_r}\vec{\nabla} + \mathbf{e_r}\frac{ie}{\hbar c}\vec{A} + \mathbf{e_{tr}}\frac{m_0c}{\hbar}\right)\tilde{\boldsymbol{\psi}} = 0.\end{gathered} \quad \text{(A 2.53)}$$

This equation describes the particle with spin 1/2 in an external electromagnetic field [21].

There is a special class of particles described by the first-order wave equation. For these particles the sedenionic Dirac-like wave equation has the following form:

$$\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}+\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\tilde{\boldsymbol{\psi}}=0\,. \qquad \text{(A 2.54)}$$

Analogously the electrically charged particle interacting with external electromagnetic field is described by the following sedenionic first-order wave equation:

$$\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{e_t}\frac{ie}{\hbar c}\varphi-\mathbf{e_r}\vec{\nabla}+\mathbf{e_r}\frac{ie}{\hbar c}\vec{A}+\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\tilde{\boldsymbol{\psi}}=0\,. \qquad \text{(A 2.55)}$$

This equation also describes the particles with spin 1/2 in an external electromagnetic field [22].

A 2.6. Generalized sedenionic equations for massive force field

The generalized sedenionic wave equation enables another interpretation. It can be considered as the equation for the force massive field. Let us consider the nonhomogeneous wave equation for the field potential with the phenomenological source of field

$$\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}+\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}+\mathbf{e_{tr}}\frac{m_0 c}{\hbar}\right)\tilde{\boldsymbol{W}}=\tilde{\boldsymbol{J}}\,. \qquad \text{(A 2.56)}$$

Here $\tilde{\boldsymbol{W}}$ is the field potential, $\tilde{\boldsymbol{J}}$ is source of field, parameter m_0 is the mass of quantum of field.

In the special case when the mass of quantum is equal to zero the equation (A 2.56) coincides with the equation for electromagnetic field in a vacuum. Indeed, choosing the potential as

$$\tilde{\boldsymbol{W}}=\mathbf{e_t}\varphi+\mathbf{e_r}\vec{A} \qquad \text{(A 2.57)}$$

and the source of electromagnetic field as

$$\tilde{\boldsymbol{J}}=-\mathbf{e_t}4\pi\rho-\mathbf{e_r}\frac{4\pi}{c}\vec{j}\,, \qquad \text{(A 2.58)}$$

we obtain the following wave equation:

$$\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(\mathbf{e_t}\varphi+\mathbf{e_r}\vec{A}\right)=-\mathbf{e_t}4\pi\rho-\mathbf{e_r}\frac{4\pi}{c}\vec{j}\ . \quad \text{(A 2.59)}$$

After the action of the first operator in the left-hand side of equation (A 2.59) we obtain

$$\begin{aligned}&\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(\mathbf{e_t}\varphi+\mathbf{e_r}\vec{A}\right)\\&=-\frac{1}{c}\frac{\partial\varphi}{\partial t}+\mathbf{e_{tr}}\frac{1}{c}\frac{\partial\vec{A}}{\partial t}+\mathbf{e_{tr}}\vec{\nabla}\varphi+\left(\vec{\nabla}\cdot\vec{A}\right)+\left[\vec{\nabla}\times\vec{A}\right].\end{aligned} \quad \text{(A 2.60)}$$

Using the sedenionic definitions of the electric and magnetic fields

$$\begin{aligned}&\vec{E}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi,\\&\vec{H}=\left[\vec{\nabla}\times\vec{A}\right]\end{aligned} \quad \text{(A 2.61)}$$

and taking into account the Lorentz gauge condition

$$\frac{1}{c}\frac{\partial\varphi}{\partial t}-\left(\vec{\nabla}\cdot\vec{A}\right)=0\ , \quad \text{(A 2.62)}$$

we can rewrite the expression (A 2.60) in the following form:

$$\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(\mathbf{e_t}\varphi+\mathbf{e_r}\vec{A}\right)=-\mathbf{e_{tr}}\vec{E}+\vec{H}\ . \quad \text{(A 2.63)}$$

Then the wave equation (A 2.59) can be represented as

$$\left(\mathbf{e_t}\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e_r}\vec{\nabla}\right)\left(-\mathbf{e_{tr}}\vec{E}+\vec{H}\right)=-\mathbf{e_t}4\pi\rho-\mathbf{e_r}\frac{4\pi}{c}\vec{j}\ . \quad \text{(A 2.64)}$$

Performing sedenionic multiplication in the left-hand side of equation (A 2.64) we get

$$
\begin{aligned}
&\mathbf{e_r}\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+\mathbf{e_t}\left(\vec{\nabla}\cdot\vec{E}\right)+\mathbf{e_t}\left[\vec{\nabla}\times\vec{E}\right]\\
&+\mathbf{e_t}\frac{1}{c}\frac{\partial\vec{H}}{\partial t}-\mathbf{e_r}\left(\vec{\nabla}\cdot\vec{H}\right)-\mathbf{e_r}\left[\vec{\nabla}\times\vec{H}\right]=-\mathbf{e_t}4\pi\rho-\mathbf{e_r}\frac{4\pi}{c}\vec{j}.
\end{aligned}
\tag{A 2.65}
$$

Separating space-time values we obtain the system of Maxwell equations in the following form:

$$
\begin{aligned}
&\mathbf{e_t}\left(\vec{\nabla}\cdot\vec{E}\right)=-\mathbf{e_t}4\pi\rho,\\
&\mathbf{e_r}\left[\vec{\nabla}\times\vec{H}\right]=\mathbf{e_r}\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+\mathbf{e_r}\frac{4\pi}{c}\vec{j},\\
&\mathbf{e_t}\left[\vec{\nabla}\times\vec{E}\right]=-\mathbf{e_t}\frac{1}{c}\frac{\partial\vec{H}}{\partial t},\\
&\mathbf{e_r}\left(\vec{\nabla}\cdot\vec{H}\right)=0.
\end{aligned}
\tag{A 2.66}
$$

The system (A 2.66) coincides with the Maxwell equations.

A 2.7. Conclusion

Algebra of sedenions is equivalent to the algebra of sedeons. In contrast to the sedeonic algebra, which uses the multiplication rules of basic elements proposed by A.Macfarlane [23], the multiplication rules for sedenionic basis elements coincide with the rules for quaternion units introduced by W.R.Hamilton [1]. There is a simple relation between these two algebras. Let as denote sedeon basis as $\mathbf{a_n}^M$ and $\mathbf{e_n}^M$ (Macfarlane rules) but sedenionic basis as $\mathbf{a_n}^H$ and $\mathbf{e_n}^H$ (Hamilton rules). Then there are the following relations:

$$
\begin{aligned}
\mathbf{a_n}^M &= i\mathbf{a_n}^H,\\
\mathbf{e_n}^M &= i\mathbf{e_n}^H.
\end{aligned}
$$

There is one disadvantage of sedenions connected with the fact that the square of the vector is a negative value. However, on the other side the sedenionic rules of cross-multiplying do not contain the imaginary unit and this leads to the considerable simplifications in the calculations. But of course, the physical results do not depend on the choice of algebra, so these two algebras are equivalent.

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